SFT Homework 2

Alexander Farren

Problem 1

I think the entries of the wavevector $\vec{k}$ are equal so that $\vec{k} = \frac{|\vec{k}|}{d} \sum_i \hat{e}_i$. As a result,

$$\vec{k} \cdot \vec{x} = \frac{|\vec{k}|}{d} \sum_j x_j$$

(Not sure if this makes it rotationally invariant, would need $|\vec{x}|^2 = \sum_j x_j^2$.)

$$\int_0^\infty dt e^{-t(k^2+1/\xi^2)} = -\frac{1}{k^2+1/\xi^2} e^{-t(k^2+1/\xi^2)}\Big|_0^\infty = \frac{1}{k^2+1/\xi^2}$$

Rest of question done in notes (p.43).

Problem 2

Derivation in notes (p.44). Interpretation:
"If we perturb the system at the origin, for a system obeying a quadratic free energy $F(\phi)$, the correlator $< \phi(\vec{x})\phi(\vec{0}) >$ responds as the solution to the original saddle point equation $0 = (-\gamma\nabla^2 + \mu^2)\tilde{m} + \alpha_4\tilde{m}^3$"

Problem 5

We start with the free energy, where $\vec{\nabla} = \frac{\partial}{\partial \vec{y}}$ and $d^d x = dx\, d^{d-1}y$,

$$F(\phi) = \frac{1}{2}\int d^d x\left[(\partial_x \phi)^2 + (\nabla^2\phi)^2 + \mu_0^2\phi^2\right]$$

If we understand Λ_0 as the maximal magnitude of the momentum k (i.e. the first component of $\vec{k}$ when $|\vec{k}| = \Lambda$) then the Fourier transform of the field in real space is given by

$$\phi(\vec{x}) = \frac{1}{(2\pi)^d}\int_0^{\Lambda} d^d k\, e^{i\vec{k}\cdot\vec{x}}\phi_{\vec{k}} = \frac{1}{(2\pi)^d}\int_0^{\Lambda_0} dk\, e^{ikx}\int_0^{\sqrt{\Lambda^2-\Lambda_0^2}} d^{d-1}q\, e^{i\vec{q}\cdot\vec{y}}\phi_{\vec{k}}$$

and the respective gradients are

$$\partial_x\phi = \frac{1}{(2\pi)^d}\int_0^{\Lambda} d^d k\,(ik)e^{i\vec{k}\cdot\vec{x}}\phi_{\vec{k}}$$

$$\nabla^2\phi = \frac{1}{(2\pi)^d}\int_0^{\Lambda} d^d k\,(-q^2)e^{i\vec{k}\cdot\vec{x}}\phi_{\vec{k}}$$

The second one can be found componentwise, with integrand o.t.f. $e^{iq_\alpha y_\alpha}$ and taking derivative $\frac{\partial}{\partial y_\beta}$. Remembering that when we have two ϕ terms multiplying, we must integrate over different momenta $\vec{k}_1 = (k_1, \vec{q}_1)$ and $\vec{k}_2 = (k_2, \vec{q}_2)$:

$$F(\phi_{\vec{k}}) = \frac{1}{2(2\pi)^{2d}}\int d^d x\int d^d k_1\int d^d k_2(-k_1k_2 + q_1^2q_2^2 + \mu_0^2)e^{i(\vec{k}_1+\vec{k}_2)\cdot\vec{x}}\phi_{\vec{k}_1}\phi_{\vec{k}_2}$$

Using the definition of the delta function $\delta^d(\vec{a}+\vec{b}) = \frac{1}{(2\pi)^d}\int d^d x\, e^{i(\vec{a}+\vec{b})\cdot\vec{x}}$, we get

$$\boxed{F(\phi_{\vec{k}}) = \frac{1}{2(2\pi)^d}\int d^d k\,(k^2 + q^4 + \mu_0^2)\,\phi_{\vec{k}}\,\phi_{-\vec{k}}}$$

We perform the scaling

$$k' = \zeta k \qquad \vec{q}\,' = \zeta^a\vec{q} \qquad \phi'_{\vec{k}'} = \zeta^{-b}\phi_{\vec{k}}$$

while imposing that the new free energy $F(\phi'_{\vec{k}'})$ have the same functional form as $F(\phi_{\vec{k}})$ with coefficients of 1 in front of k'^2 and q'^4. We get $d^d k = d^d k'\zeta^{-1-(d-1)a}$ with each dq contributing ζ^{-a},

$$\begin{aligned}F(\phi'_{\vec{k}'}) &= \frac{1}{2(2\pi)^d}\int d^d k\zeta^{-1+(1-d)a}\,(\zeta^{-2}k'^2 + \zeta^{-4a}q'^4 + \mu_0^2)\zeta^{2b}\phi'_{\vec{k}'}\,\phi'_{\vec{k}'}\\ &= \frac{1}{2(2\pi)^d}\int d^d k\zeta^{2b-1-da-3a}\,(\zeta^{4a-2}k'^2 + q'^4 + \zeta^{4a}\mu_0^2)\phi'_{\vec{k}'}\,\phi'_{\vec{k}'}\end{aligned}$$

$$\Longrightarrow\ 4a - 2 = 0 \Longrightarrow \boxed{a = 1/2} \Longrightarrow 2b - 1 - d/2 - 3/2 = 0 \Longrightarrow \boxed{b = (5+d)/4}$$

$$\Longrightarrow \boxed{\mu^2(\zeta) = \zeta^2\mu_0^2}$$

Returning to real space

$$F(\phi) = \frac{1}{2}\int d^dx\big[(\partial_x\phi)^2 + (\nabla^2\phi)^2 + \mu_0^2\phi^2\big]$$

This time the scaling is the opposite

$$x' = x/\zeta \qquad \vec{y}' = \zeta^{-a}\vec{y} \qquad \phi'(\vec{x}') = \zeta^{\Delta_\phi}\phi(\vec{x})$$

which gives $\partial_x = \zeta^{-1}\partial_{x'}$ and $\vec{\nabla}' = \zeta^{-a}\vec{\nabla}$. Thus

$$F(\phi'(\vec{x}')) = \frac{1}{2}\int d^dx'\zeta^{1+(d-1)a-2-2\Delta_\phi}\big[(\partial_{x'}\phi')^2 + (\nabla'^2\phi')^2 + \mu(\zeta)^2\phi'^2\big]$$

This means $\boxed{\Delta_\phi = (d-3)/4}$ since $a = 1/2$. Next we look at $g_n = \zeta^{\Delta_{g_n}} g_{0,n}$:

$$\int d^dx\ g_{0,n}\phi^{2n} = \int d^dx'\zeta^{1+(d-1)a-2n\Delta_\phi-\Delta_{g_n}}g_n\phi'^{2n}$$

$$\implies \Delta_{g_n} = \frac{1}{2}(2+d-1-dn+3n) = \boxed{\frac{1}{2}(1+3n+d(1-n))}$$

If we are looking at $g_4(\zeta)$ then $n = 2$ and we get

$$\Delta_{g_4} = \frac{1}{2}(7-d)\begin{cases}\Delta_{g_4} < 0 \quad d > 7 \implies & \text{vanishes after many RG flows thus } \textit{irrelevant} \\ \Delta_{g_4} > 0 \quad d < 7 \implies & \text{grows with each RG flow thus } \textit{relevant} \quad \square.\end{cases}$$

Problem 6

We start with free energy

$$F(\psi, A_i) = \int d^dx\ \big[\frac{1}{4}F_{ij}F^{ij} + |\partial_i\psi - ieA_i\psi|^2 + \mu^2|\psi|^2\big]$$

Applying the rescalings

$$x_i' = x_i/\zeta \implies \partial_i' = \zeta\partial_i \qquad d^dx = d^dx'\zeta^d$$
$$A_i' = \zeta^{\Delta_A}A_i \qquad \psi'(x_i') = \zeta^{\Delta_\psi}\psi(x_i)$$

The first term scales as

$$d^dx\ F_{ij}F^{ij} = \zeta^{d-2-2\Delta_A}d^dx'\ F'_{ij}F'^{ij}$$

The second term scales as

$$\begin{aligned} d^dx\ |\partial_i\psi - ieA_i\psi|^2 &= d^dx\big[|\partial_i\psi|^2 + \text{ mixed terms } + |eA_i\psi|^2\big] \\ &= \zeta^d d^dx'\big[\zeta^{-2-2\Delta_\psi}|\partial_i'\psi'|^2 + \ldots + \zeta^{-2\Delta_A-2\Delta_\psi}|eA_i'\psi'|^2\big]\end{aligned}$$

Requiring that the gradient terms ($F_{ij}F^{ij}$ and $\partial\psi$) remain canonically normalised,

$$d-2-2\Delta_A = 0 \qquad d-2-2\Delta_\psi = 0$$
$$\implies 2\Delta_A = 2\Delta_\psi = d-2$$

This tells us that the interaction coupling scaling dimension is

$$d - 2\Delta_A - 2\Delta_\psi = 4-d$$

which is relevant for $d > d_c$, irrelevant for $d < d_c$ where $\boxed{d_c = 4}$