SFT Homework 1

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Problem 1

We have

$$Z = \sum_{s_1=\pm1} \dots \sum_{s_N=\pm1} \prod_{i=1}^{N} \exp(\beta J s_i s_{i+1} + \frac{1}{2}\beta B(s_i + s_{i+1})) = \sum_{s_1=\pm1} \dots \sum_{s_N=\pm1} \prod_{i=1}^{N} T_{s_i,s_{i+1}}$$

If we consider s_i/s_{i+1} as the index denoting T's row/column, then

$$Z = \sum_{s_1=\pm1} \sum_{s_2=\pm1} \dots \sum_{s_N=\pm1} T_{s_1,s_2} T_{s_2,s_3} \dots T_{s_N,s_1} = \sum_{i=\pm} \sum_{j=\pm1} \sum_{k=\pm1} \dots \sum_{l=\pm1} T_{ij} T_{jk} \dots T_{li}$$

which is just

$$Z = \sum_{i=\pm1} (T^N)_{ii} = \mathrm{tr}(T^N) \quad \square.$$

In matrix form, with eigenvalues $\lambda_\pm$

$$T = \begin{pmatrix} e^{\beta J - \beta B} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J + \beta B} \end{pmatrix} \implies \det(T - \lambda_\pm I) = 0$$

$$\det(T - \lambda_\pm I) = \lambda_\pm^2 - e^{\beta J}(e^{\beta B} + e^{-\beta B})\lambda_\pm + (e^{2\beta J} - e^{-2\beta J}) = 0$$

$$\lambda_\pm = e^{\beta J}\cosh(\beta B) \pm \sqrt{e^{2\beta J}\cosh^2(\beta B) - 2\sinh^2(2\beta J)} \quad \square.$$

Having found the eigenvalues of T, let M be the matrix which diagonalises T such that $D = MTM^{-1}$ where $D = \mathrm{diag}(\lambda_-, \lambda_+)$. Since $\mathrm{tr}(ABC) = \mathrm{tr}(BCA)$, we have

$$Z = \mathrm{tr}(T^N) = \mathrm{tr}(M^{-1}DMM^{-1}D \dots M^{-1}DM) = \mathrm{tr}(M^{-1}D^N M) = \mathrm{tr}(D^N) = \lambda_+^N + \lambda_-^N$$

But $\lambda_+ > \lambda_-$ since $e^{\beta J}\cosh(\beta B) > 0$ for real βJ and βB, then $\lim_{N\to\infty} Z \approx \lambda_+^N$. The magnetisation is

$$\tilde{m} = \frac{1}{N\beta}\frac{\partial}{\partial B}\ln Z = \frac{1}{\lambda_+\beta}\frac{\partial \lambda_+}{\partial B}$$

Note that $\lambda_+\big|_{B=0} = 2\cosh\beta J \neq 0$ for real βJ. We must find

$$\frac{\partial \lambda_+}{\partial B} = \beta(e^{\beta J}\sinh(\beta B) + e^{2\beta J}\cosh(\beta B)\sinh(\beta B)[e^{2\beta J}\cosh^2(\beta B) - 2\sinh(2\beta J)]^{-1/2})$$

$$\implies \tilde{m}\big|_{B=0} = \frac{1}{\lambda_+\beta}\frac{\partial \lambda_+}{\partial B}\Big|_{B=0} = 0 \quad \forall \beta J \in \mathbb{R}$$

If the magnetisation is always 0, then it is constant and it along with its derivatives are not discontinuous in β. This is synonymous with there being no phases transitions as a function β or temperature T.

Problem 2

Given the approximation $s_i s_j \approx \tilde{m}(s_i + s_j) - \tilde{m}^2$, with q the number of nearest neighbour pairs per site, and $<ij>$ is the set of nearest neighbour pairs (not sites),

$$\begin{aligned} E &= -B\sum_{i=1}^{N} s_i - J\sum_{<ij>} s_i s_j = -B\sum_{i=1}^{N} s_i - \frac{1}{2}Jq\tilde{m}\sum_{i,j=1}^{N}(s_i + s_j) + \frac{1}{2}NqJ\tilde{m}^2 \\ &= -(Jq\tilde{m} + B)\sum_{i=1}^{N} s_i + \frac{1}{2}NqJ\tilde{m}^2 \\ \implies Z &= \sum_{\{s_i\}} e^{-\beta E[s_i]} = e^{-\beta\frac{1}{2}NqJ\tilde{m}^2}\sum_{\{s_i\}} e^{\beta(Jq\tilde{m}+B)\sum_i s_i} = e^{-\beta\frac{1}{2}NqJ\tilde{m}^2}\sum_{s_1=\pm1}\cdots\sum_{s_N=\pm1}\prod_{i=1}^{N} e^{\beta(Jq\tilde{m}+B)s_i} \\ &= e^{-\beta\frac{1}{2}NqJ\tilde{m}^2}\left(e^{\beta(Jq\tilde{m}+B)} + e^{-\beta(Jq\tilde{m}+B)}\right)^N =^{-\beta\frac{1}{2}NqJ\tilde{m}^2} 2^N\cosh^N(\beta(Jq\tilde{m} + B)) \quad \square. \end{aligned}$$

Finding the equilibrium magnetisation:

$$\begin{aligned} \tilde{m} &= \frac{1}{N\beta}\frac{\partial}{\partial B}\ln Z = \frac{1}{\beta}\frac{\partial}{\partial B}\ln\cosh(\beta(B + Jq\tilde{m})) \\ &= \frac{\beta\sinh(\beta(B + Jq\tilde{m}))}{\beta\cosh(\beta(B + Jq\tilde{m}))} = \tanh(\beta(B + Jq\tilde{m})) \quad \square. \end{aligned} \tag{1}$$

For $B = 0$, we have $\tilde{m} = \tanh(Jq\tilde{m})$. Note that $\beta Jq = \frac{T_c}{T}$ such that $T < T_c \implies \beta Jq > 1$ and vice versa.

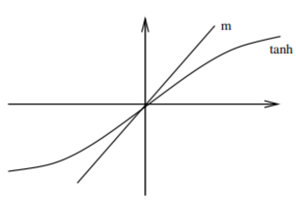


Figure 43: $\tanh(Jqm\beta)$ for $Jq\beta < 1$

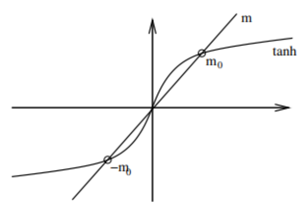


Figure 44: $\tanh(Jqm\beta)$ for $Jq\beta > 1$

For $T < T_c$ there are two solutions $\tilde{m} = \pm m_0$. For $T > T_c$ there is only one solution $\tilde{m}$. In particular as $T \to \infty$, $\beta \to 0$ and $\tilde{m} \to 0$ by the consistency equation (1).

Problem 3

By completing the square and remembering the Gaussian integral $\int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$,

$$\int_{-\infty}^{\infty} e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}} \implies e^{\frac{\beta J\alpha^2}{2N}} = \sqrt{\frac{N\beta J}{2\pi}}\int_{-\infty}^{\infty} dx\; e^{-\frac{N\beta J}{2}x^2 + \alpha\beta Jx}$$

Starting with $Z = \sum_{\{s_i\}} e^{-\beta E[s_i]}$ and letting $k = \sum_{i=1}^{N} s_i$ we can write

$$\begin{aligned} Z &= \sum_k e^{\beta Bk + \frac{\beta J}{2N}k^2} = \sum_k e^{\beta Bk}\sqrt{\frac{N\beta J}{2\pi}}\int_{-\infty}^{\infty} dx\; e^{-\frac{N\beta J}{2}x^2 + k\beta Jx} \\ &= \sqrt{\frac{N\beta J}{2\pi}}\int_{-\infty}^{\infty} dx\; e^{-\frac{N\beta J}{2}x^2}\sum_k e^{k\beta(B+Jx)} \end{aligned}$$

As shown in Problem 2,

$$\sum_k e^{\beta(B+Jx)k} = \sum_{\{s_i\}} e^{\beta(B+Jx)\sum_i s_i} = 2^N \cosh^N(\beta(B+Jx))$$

Thus we get

$$\begin{aligned} Z &= \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-\frac{N\beta J}{2}x^2} 2^N \cosh^N(\beta(B+Jx)) = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-\frac{N\beta J}{2}x^2 + N\ln(2\cosh(\beta(B+Jx)))} \\ &= \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-NS(x)} \qquad \text{where} \quad S(x) = \frac{\beta J}{2}x^2 - \ln(2\cosh(\beta(B+Jx))) \quad \square. \end{aligned}$$

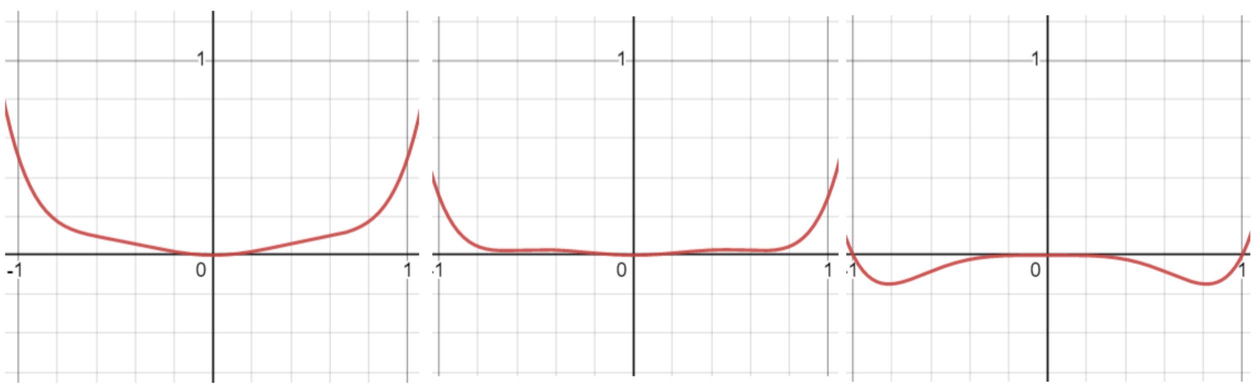
Taking the derivative and setting to 0 yields

$$\frac{dS}{dx}\Big|_{x^*} = \beta J x^* - \beta J \tanh(\beta(B+Jx^*)) = 0 \implies x^* = \tanh(\beta B + \beta J x^*)$$

In the limit of large N: $Z \approx e^{-N\beta f(\tilde{m})}$. If we make the identification $S(x) = \beta f(x)$ (where $f(m)$ is the effective free energy per unit spin), then S achieves a minimum whenever f does (i.e. $x^* = \tilde{m}$). This explains why they follow the same self-consistency equation (1) up to a factor.

Problem 4

Below are sketches on Desmos for $\alpha_6 = -\alpha_4 = 1$ and $\alpha_2 = 0.5, 0.3, 0$ from left to right. The system



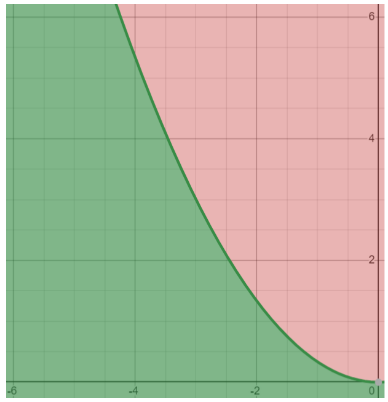
$$f(m) = \alpha_2 m^2 + \alpha_4 m^4 + \alpha_6 m^6$$

undergoes a first order phase transition when the first derivative of the free energy $f(m)$ is discontinuous. Equivalently, $\tilde{m}$ which minimises $f(m)$ is discontinuous. The phase transition occurs when $\tilde{m}$ jumps between two values.

$$\frac{\partial f}{\partial m}\Big|_{\tilde{m}} = 2\alpha_2\tilde{m} + 4\alpha_4\tilde{m}^3 + 6\alpha_6\tilde{m}^5 = 0$$

Either $\tilde{m} = 0$ or $2\alpha_2 + 4\alpha_4\tilde{m}^2 + 6\alpha_6\tilde{m}^4 = 0 \implies \tilde{m}^2 = -\frac{\alpha_4}{3\alpha_6} \pm \sqrt{(\frac{\alpha_4}{3\alpha_6})^2 - \frac{\alpha_2}{3\alpha_6}} \equiv m_\pm^2$.

The solutions $m_\pm$ only exist when the discriminant $(\frac{\alpha_4}{3\alpha_6})^2 - \frac{\alpha_2}{3\alpha_6}$ is non-negative (and when $m_\pm^2 \geq 0$). This first occurs when the discriminant is zero, or when $\alpha_2 = \frac{\alpha_4^2}{3\alpha_6}$. For the values defining the above graphs, this would be $\alpha_2 = 0.33$. However these correspond to non-zero local minima and not the 'dips' which appear after $\alpha_2 = 0.25 = \frac{\alpha_4^2}{4\alpha_6}$(?).



Phase diagram in $\alpha_4 - \alpha_2$ plane. The red region is when the discriminant is negative, and $\tilde{m} = 0$ whereas in the green region $\tilde{m} = m_\pm$. The phase boundary is $\alpha_2 = \frac{1}{3\alpha_6}\alpha_4^2$.

The jump in magnetisation between 0 and $m_\pm$ happens when $\alpha_2 = \frac{1}{3\alpha_6}\alpha_4^2 \implies m_\pm^2 = -\frac{\alpha_2}{\alpha_4}$. Thus the jump is $\boxed{m_0 = \sqrt{-\alpha_2/\alpha_4}}$ where $\tilde{m} = 0 \to \tilde{m} = \pm m_0$.

Finding the critical exponents for $\alpha_4 = 0$ ($B = 0$ for α, β, whereas $B \neq 0$ for γ, δ):

$$c \sim |T - T_c|^{-\alpha} \qquad \tilde{m} \sim |T - T_c|^{\beta} \qquad \chi \sim |T - T_c|^{-\gamma} \qquad \tilde{m} \sim B^{\frac{1}{\delta}}$$

We have already found $\tilde{m}^2 = \sqrt{-\frac{\alpha_2}{3\alpha_6}} \implies \boxed{\beta = 1/4}$. Using Mathematica $\boxed{\alpha = 1/2}$.

```
m̃[T_] = ((Tc - T)/(3 α6))^(1/4); (*roots m̃ *)
F[β_] = β (1 / β - Tc) m̃[1 / β]^2 + α6 β m̃[1 / β]^6; (* βf(m̃(β)) *)
T^(-2) * F''[1 / T] // FullSimplify (*c=β^2 ∂^2/∂β^2 [ βf(m̃(β)) ] *)

T Sqrt[(-T+Tc)/α6]
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Sqrt[3] (2 T - 2 Tc)
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Near the critical point, for $B \neq 0$, $f(m) \approx -Bm + \alpha_6 m^6 \implies \tilde{m} \sim B^{\frac{1}{5}} \implies \boxed{\delta = 5}$. It looks like $\boxed{\gamma = 1}$ as in the notes.

Problem 5

I think this is the same analysis as in the notes (starting page 12) with the added possibility of a phase so that $\tilde{\psi} \sim \tilde{m}e^{i\phi}$ minimises the free energy. Maybe spontaneous symmetry breaking is then related to complex conjugating as well as the usual $\mathbb{Z}_2$ symmetry.

Problem 6 (by Luke Hodgkiss)

Low T

Thermal fluctuations negligible

- $J > 0$ so spins align in a ferromagnetic ordering (same dirxn)

i) $g < 0 \Rightarrow S_i^z$ term is negative so energy is minimized when most of spin in S^z component (i.e maximized). This is the Ising model type ordering

ii) $g = 0 \Rightarrow$ neither phase preferred so both phases coexist

iii) $g > 0 \Rightarrow S^x - S^y$ term is negative so energy is minimized when spin lies in x-y plane. This is the plane rotator

High T

Thermal fluctuations dominate, ordering destroyed

dont really get this fully, solution arent very detailed

i) $g < 0$ $S^x, S^y \approx 0$ recover Ising model so increasing T causes a 2nd order phase transition

ii) $g > 0$ $S^z \approx 0$ recover plane rotator, 2nd order phase transition

Problem 7

We are given $\psi(x) = \frac{1}{V} e^{ikx} \psi_k = a_k e^{2ikx}$ where I have defined $a_k = \frac{A_k}{V}$. Thus

$$\psi'(x) = 2ik\psi(x) \qquad \psi''(x) = -4k^2\psi(x)$$

which tells us

$$\begin{aligned} F &= \int dx(\alpha_2|\psi(x)|^2 + \alpha_4|\psi(x)|^4 - \gamma|\psi'(x)|^2 + \kappa|\psi''(x)|^2) \\ &= \int dx|a_k|^2(\alpha_2 + \alpha_4|a_k|^2 - 4k^2\gamma + 16k^4\kappa) \end{aligned}$$

The value of $\tilde{k}$ which minimises F is given by $\frac{\delta F}{\delta k}\Big|_{\tilde{k}} = 0$. Thus, assuming that a_k is a constant and remembering that $k = \pm k_0 \implies \tilde{k} = \pm k_0$,

$$\frac{\delta F}{\delta k}\Big|_{\tilde{k}} = -8\tilde{k}\gamma + 64\tilde{k}^3\kappa = 0 \implies \tilde{k} = 0 \text{ or } \pm\sqrt{\frac{\gamma}{8\kappa}} \implies \boxed{k_0 = 0 \text{ or } \sqrt{\frac{\gamma}{8\kappa}}}$$

No idea how to relate α_2 to the other constants if the (discontinuous?) order parameter is k? If the order parameter is $\psi(x)$ then you need to do $\frac{\delta\psi\psi*}{\delta\psi}$ etc...?

Problem 8

We have $f(m) = \alpha_2 m^2 + \alpha_{2n} m^{2n}$ so that the equilibrium magnetisations are

$$\tilde{m} = 0 \;\text{ or }\; \tilde{m} = \left(\frac{T_c - T}{\alpha_{2n} n}\right)^{\frac{1}{2n-2}} \implies \boxed{\beta^* = \frac{1}{2n-2}}$$

Here β^* denotes the critical exponent while $\beta = 1/T$. Using Mathematica I showed $\alpha = 1 - 2\beta^*$. □

$$\tilde{m}[x\_] = \left(\frac{Tc - x}{\alpha_{2n}\, n}\right)^{\frac{1}{2n-2}}; \;\; (*\text{roots } \tilde{m}\ *)$$

$$F[\beta\_] = \beta\, (1/\beta - Tc)\, \tilde{m}[1/\beta]\text{^}2 + \alpha_{2n}\, \beta\, \tilde{m}[1/\beta]\text{^}(2\, n); \;\; (*\ \beta f(\tilde{m}(\beta))\ *)$$

$$T\text{^}(-2) * F''[1/T] \text{ // FullSimplify } \;\; \left(*c = \beta^2 \frac{\partial^2(\beta f(\tilde{m}(\beta)))}{\partial\beta^2}\ *\right)$$

$$\frac{n\, T\left((T - Tc)\left(\frac{-T+Tc}{n\, \alpha_{2n}}\right)^{\frac{1}{-1+n}} + \left(\left(\frac{-T+Tc}{n\, \alpha_{2n}}\right)^{\frac{1}{2\,(-1+n)}}\right)^{2n} \alpha_{2n}\right)}{(-1+n)^2\, (T - Tc)^2}$$

$$\frac{n\, T\left((T - Tc)\left(\frac{-T+Tc}{\alpha_{2n}\, n}\right)^{\frac{1}{n-1}} + a2\left(\frac{-T+Tc}{\alpha_{2n}\, n}\right)^{\frac{n}{n-1}}\right)}{(-1+n)^2\, (T - Tc)^2} \text{ /. } T \to Tc - y\ \&\ y \to x\, \alpha_{2n}\, n \text{ // FullSimplify}$$

$$\frac{n\, (Tc - y)\left(-y\left(\frac{y}{\alpha_{2n}\, n}\right)^{\frac{1}{-1+n}} + a2\left(\frac{y}{\alpha_{2n}\, n}\right)^{\frac{n}{-1+n}}\right)}{(-1+n)^2\, y^2} \text{ /. } y \to x\, \alpha_{2n}\, n \text{ // FullSimplify}$$

$$\frac{x^{-1+\frac{1}{-1+n}}\, (-a2 + n\, \alpha_{2n})\, (-Tc + n\, x\, \alpha_{2n})}{(-1+n)^2\, n\, \alpha_{2n}^2} \;\; \left(*\text{here } x \sim (Tc - T) \text{ so that } \alpha = 1 - \frac{1}{1-n} = 1 - 2\beta^*\ *\right)$$