Bootstrapping amplitudes

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Part III talk

6 December 2024

Bootstrapping amplitudes

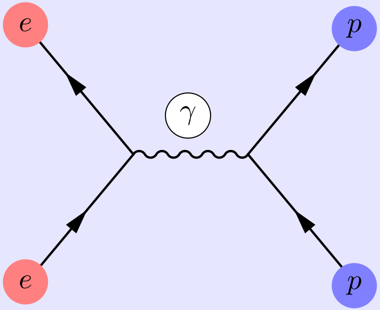
1. Introduction
2. Parke-Taylor formula
3. Loop amplitudes

1 Introduction

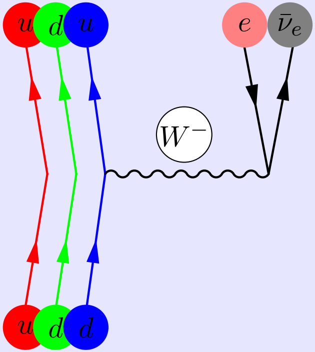
Introduction

Motivation of particle physics: identify what makes up the world, explain physical phenomena from particle interactions (3/4 forces explained by quantum field theory).

electromagnetic



weak



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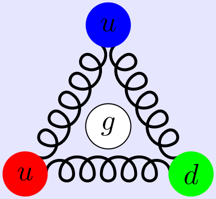


Introduction

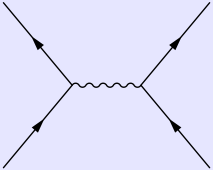
Quantum field theory (**without Feynman diagrams**):

$$
\begin{aligned}
\text{Physical system} &\rightarrow \quad \mathcal{L}(\phi, \dot{\phi}, ...) && \text{Lagrangian} \\
\mathcal{L}(\phi, \dot{\phi}, ...) &\rightarrow \quad \mathcal{H}(\phi, \pi, ...) && \text{Hamiltonian} \\
\\
\mathcal{H}(\phi, \pi, ...) &\rightarrow \quad \hat{\mathcal{H}}(\hat{\phi}, \hat{\pi}, ...) && \text{Quantisation} \\
\{\phi(\mathbf{x}), \pi(\mathbf{y})\}_{\text{Poisson}} = \delta^{(3)}(\mathbf{x}-\mathbf{y}) &\rightarrow \quad [\hat{\phi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = \mathrm{i}\delta^{(3)}(\mathbf{x}-\mathbf{y}) && \\
\phi(t) &\rightarrow \quad \langle 0|T\{\hat{\phi}(x)\hat{\phi}(y)\}|0\rangle && \\
\\
\text{LSZ, Schwinger-Dyson, Wick's} &\rightarrow \quad \langle \mathbf{k}_1\mathbf{k}_2...|\mathcal{S}|\mathbf{p}_1\mathbf{p}_2...\rangle && \text{Scattering} \\
\mathcal{S} &\rightarrow \quad \mathcal{A} && \\
\mathcal{A} &\rightarrow \quad d\sigma/d\Omega, \Gamma, ... &&
\end{aligned}
$$

Introduction

Quantum field theory (**with Feynman diagrams**):

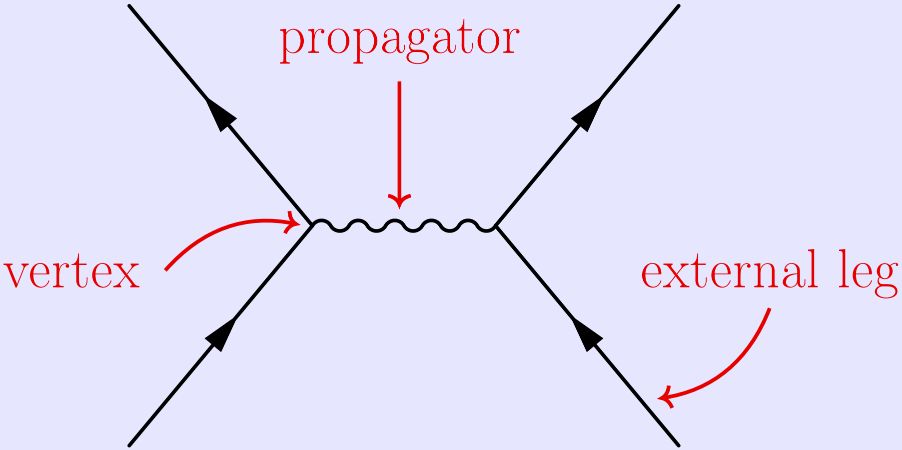
$$\mathcal{L} \to$$



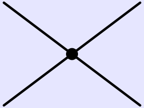
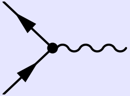
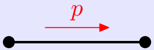
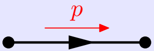
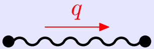
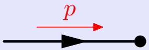
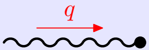
$$\to \mathcal{M} \to \frac{d\sigma}{d\Omega}, \Gamma, \ldots$$

Given Feynman rules, can interpret diagrams to get $\mathcal{M}$. Hooray, everything is great!
Or is it?...

Feynman diagrams



Feynman diagrams

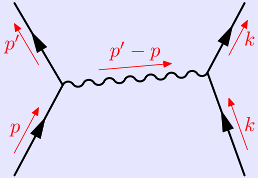
ϕ^4 **theory**		**Quantum ElectroDynamics**	
$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$		$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F^2 + \bar{\Psi}\left(\mathrm{i}\gamma^\mu D_\mu - m\mathbb{1}\right)\Psi$	
	$= -\mathrm{i}\lambda$		$= -\mathrm{i}e\gamma^\mu$
	$= \dfrac{\mathrm{i}}{p^2 - m^2}$		$= \dfrac{\mathrm{i}\left(\gamma^\alpha p_\alpha + m\mathbb{1}\right)}{p^2 - m^2}$
			$= \dfrac{-\mathrm{i}\eta_{\mu\nu}}{q^2}$
	$= 1$		$= u^s(p)$
			$= \epsilon^\mu(q)$

Feynman diagrams

Additional rules:

- Impose momentum conservation at each vertex.
- Integrate over undetermined loop momentum.
- Divide by symmetry factor.
- Account for fermion minus sign.

Example:



$$\sim \mathrm{i}\mathcal{A} = \bar{u}^{r'}(k')(-\mathrm{i}e\gamma^{\mu})u^{r}(k)\left(\frac{-\mathrm{i}\eta_{\mu\nu}}{(p'-p)^2}\right)\bar{u}^{s'}(p')(-\mathrm{i}e\gamma^{\nu})u^{s}(p),$$

2 Parke-Taylor formula

Bootstrap

"Pull yourself up by your bootstraps"

Bootstrap

Input principles

$$S[\Psi, A_\mu, \phi] = \int d^4x \; \mathcal{L}(\Psi, A_\mu, \phi)$$

Scattering amplitude $\mathcal{A}_n$

Bootstrap toolkit

Dimensional	Mass dimension $[\mathcal{A}_n] = D + n - Dn/2$.
Rational	$\mathcal{A}_n$ is a finite polynomial in momenta, polarisations.
Crossing symmetry	$\mathcal{A}_n(e_L^-(p)$ going in$) \leftrightarrow \mathcal{A}_n(e_R^+(-p)$ going out$)$.
Lorentz invariance	$\mathcal{A}_n = \mathcal{A}_n(\{p_i \cdot p_j\}, \{p_i \cdot \varepsilon_j\}, \{\varepsilon_i \cdot \varepsilon_j\})$.
Gauge invariance	Feynman diagrams may depend on gauge, but not $\mathcal{A}_n$.
Spin-statistics	For fermions/bosons $\mathcal{A}_n(12\ldots n) = \pm\mathcal{A}_n(21\ldots n)$.
Analyticity	At most simple poles in Mandelstam invariants, e.g. $\mathcal{A} \neq s^{-2}(\cdots)$.
Soft theorems	$\lim_{p\to 0} \mathcal{A}_n(\ldots p \ldots) \sim p^\sigma$.

Spinor-helicity formalism

Complexified Lorentz group is $SO(3,1)_{\mathbb{C}} \cong SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ so that, in chiral rep,

$$\gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu) \\ (\bar{\sigma}^\mu) & 0 \end{pmatrix}$$

$$(\gamma^\mu)_{a\dot{b}} = (\sigma^\mu)_{a\dot{b}}, \qquad (\gamma^\mu)^{\dot{a}b} = (\bar{\sigma}^\mu)^{\dot{a}b}$$

with $a = 1, 2$, $\dot{a} = \dot{1}, \dot{2}$, vectors transform as

$$\not{p} = p_\mu \gamma^\mu = \begin{pmatrix} 0 & p_{a\dot{b}} \\ p^{\dot{a}b} & 0 \end{pmatrix}$$

$$p_{a\dot{b}} = p_\mu (\sigma^\mu)_{a\dot{b}}, \qquad p^{\dot{a}b} = p_\mu (\bar{\sigma}^\mu)^{\dot{a}b}.$$

For massless particle, $p_{a\dot{a}}$ has rank 1 so that

$$\det(p_{a\dot{a}}) = p_\mu p^\mu = 0 \implies p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}.$$

From now: restrict discussion to massless particles, or equivalently high-energy regime.

Spinor-helicity formalism

We now express momenta in terms of

- **angle brackets** $\langle ij\rangle = \epsilon^{ab}\lambda_{i,a}\lambda_{j,b}$,
- **square brackets** $[ij] = \epsilon^{\dot{a}\dot{b}}\tilde{\lambda}_{i,\dot{a}}\tilde{\lambda}_{j,\dot{b}}$.

For example,

$$\langle 12\rangle[12] = \epsilon^{ab}\lambda_{1,a}\lambda_{2,b}\epsilon^{\dot{a}\dot{b}}\tilde{\lambda}_{1,\dot{a}}\tilde{\lambda}_{2,\dot{b}} = \ldots = 2p_1\cdot p_2 = (p_1+p_2)^2 = s_{12}.$$

Example in scalar Yukawa:

$$\begin{aligned}\mathcal{A}_4(\phi\bar{f}^+f^-\phi) &= -g^2\frac{\langle 3|p_1+p_2|2]}{(p_1+p_2)^2} + (1\leftrightarrow 4)\\ &= -g^2\frac{-\langle 31\rangle[12]}{\langle 12\rangle[12]} + (1\leftrightarrow 4)\\ &= -g^2\left(\frac{\langle 13\rangle}{\langle 12\rangle}+\frac{\langle 34\rangle}{\langle 24\rangle}\right).\end{aligned}$$

Thus

$$|\mathcal{A}_4|^2 = \left(\frac{\langle 13\rangle}{\langle 12\rangle}+\frac{\langle 34\rangle}{\langle 24\rangle}\right)\left(\frac{[13]}{[12]}+\frac{[34]}{[24]}\right) = \ldots = g^4\frac{(s-t)^2}{st}.$$

Little group

Subgroup of $SO(3,1)_{\mathbb{C}}$ which leaves momenta p_i invariant:

$$\lambda_{i,a} \to t_i \lambda_{i,a}, \qquad \tilde{\lambda}_{i,\dot{a}} \to t_i^{-1} \tilde{\lambda}_{i,\dot{a}}.$$

If $p_i \in \mathbb{R}^{3,1}$ then $|t_i| = 1$, otherwise just $t_i \in \mathbb{C}$. What about $\varepsilon^\mu_\pm(p)$?
Polarisations in spinor-helicity notation:

$$\varepsilon^+_{a\dot{a}}(p) = \frac{\eta_a \tilde{\lambda}_{\dot{a}}}{\langle \eta \lambda \rangle}, \qquad \varepsilon^-_{a\dot{a}}(p) = \frac{\lambda_a \tilde{\eta}_{\dot{a}}}{[\tilde{\eta} \tilde{\lambda}]}.$$

Freedom to choose $\eta \neq \lambda$ since Ward identity identifies

$$\varepsilon^\mu(p) \longleftrightarrow \varepsilon^\mu(p) + c p^\mu$$

and $\eta_a \to \alpha \eta_a + \beta \lambda_a$ induces

$$\varepsilon^+_{a\dot{a}}(p) \longrightarrow \varepsilon^+_{a\dot{a}}(p) + \frac{\beta}{\alpha \langle \eta \lambda \rangle} p_{a\dot{a}}.$$

Little group scaling

h	polarisation	LG weight
0	1	0
$\pm 1/2$	$u_\pm \sim \lambda$	∓ 1
± 1	$\varepsilon^\mu_\pm$	∓ 2
$\pm 3/2$	$u_\pm \varepsilon^\mu_\pm$	∓ 3
± 2	$\varepsilon^\mu_\pm \varepsilon^\nu_\pm$	∓ 4

$$\mathcal{A}_n(1^{h_1} 2^{h_2} \ldots n^{h_n}) \xrightarrow{\text{Lg}} \left(\prod_i t_i^{-2h_i} \right) \mathcal{A}_n(1^{h_1} 2^{h_2} \ldots n^{h_n})$$

Bootstrap in action

Q: What is $\mathcal{A}_3(1^{h_1}2^{h_2}3^{h_3})$?

A: Start by noticing $\langle 12\rangle[12] = (p_1+p_2)^2 = (-p_3)^2 = 0$ so either

- $\langle 12\rangle \neq 0$ but $\langle 12\rangle[23] = -\langle 11\rangle[13] - \langle 13\rangle[33] = 0$
$$\implies [12] = [23] = [31] = 0,$$
- $[12] \neq 0$ but $\langle 31\rangle[12] = -\langle 32\rangle[22] - \langle 33\rangle[32] = 0$
$$\implies \langle 12\rangle = \langle 23\rangle = \langle 31\rangle = 0.$$

But $\mathcal{A}_3$ must be a rational function of $\langle ij\rangle$, $[ij]$ so

$$\mathcal{A}_3(1^{h_1}2^{h_2}3^{h_3}) = g \times \begin{cases} \langle 12\rangle^{n_3}\langle 23\rangle^{n_1}\langle 31\rangle^{n_2} & \text{and } [12] = [23] = [31] = 0, \\ [12]^{n_3}[23]^{n_1}[31]^{n_2} & \text{and } \langle 12\rangle = \langle 23\rangle = \langle 31\rangle = 0. \end{cases}$$

Bootstrap in action

$$\mathcal{A}_3(1^{h_1}2^{h_2}3^{h_3}) = g\langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h_1-h_3}$$

Helicity amplitudes

The following tree amplitudes for n gluon scattering vanish:

$$\mathcal{A}_n(1^+2^+\dots n^+) = 0,$$
$$\mathcal{A}_n(1^-2^+\dots n^+) = 0.$$

Maximally helicity violating (MHV) amplitudes are given by:

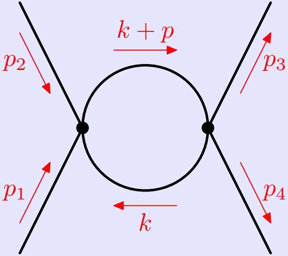
Parke-Taylor formula

$$\mathcal{A}_n(1^+2^+\dots i^-\dots j^-\dots n^+) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$

3 Loop amplitudes

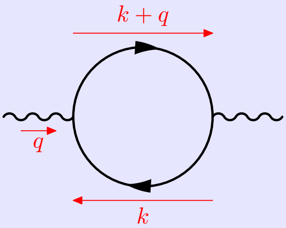
One-loop integrals

In order to isolate divergences, use dimensional regularisation with $D = 4 - 2\epsilon$:



$$\propto \int d^D k \, \frac{\mathrm{i}}{(k+p)^2 - m^2} \frac{\mathrm{i}}{k^2 - m^2} \sim \left[\frac{1}{\epsilon} - \gamma_E - \log\left(\frac{\Delta}{4\pi}\right) + \mathcal{O}(\epsilon) \right]$$

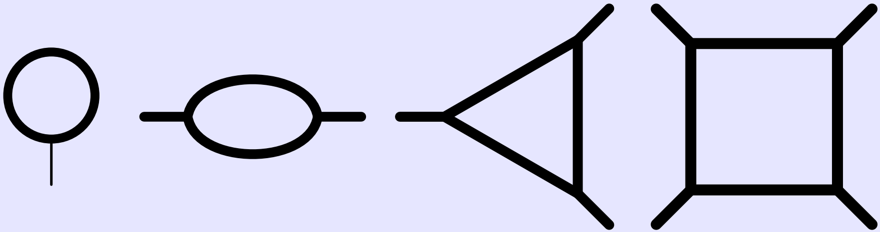
$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right)$$



$$\propto \int d^D k \, \frac{\mathrm{tr}\left[\gamma^\mu \left(\not{k} + m\right) \gamma^\nu \left(\not{k} + \not{q} + m\right)\right]}{\left(k^2 - m^2\right)\left((k+q)^2 - m^2\right)} \sim \left(q^2 \eta^{\mu\nu} - q^\mu q^\nu\right) \left[\dots \right]$$

One-loop integrals

We can express any one-loop Feynman integral in terms of the first four scalar integrals



Tadpole Bubble Triangle Box

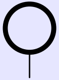
which correspond to a basis of integrals called master integrals

$$I_n^D\left(\{p_i \cdot p_j\}; m_i^2; \epsilon\right) \equiv e^{\gamma_E \epsilon} \int \frac{d^D k}{\mathrm{i}\pi^{D/2}} \frac{1}{(k-p_1)^2 - m_1^2} \frac{1}{(k-p_1-p_2)^2 - m_2^2} \cdots$$

and ϵ is the dimensional regularisation parameter such that $D = 4 - 2\epsilon$.

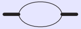
One-loop integrals

If $D = 2\left\lceil \frac{n}{2} \right\rceil - 2\epsilon$, it turns out that all one-loop Feynman integrals are basically logarithms!



$$= -e^{\gamma_E \epsilon} \left(m^2\right)^{-\epsilon} \Gamma\left(\epsilon\right)$$

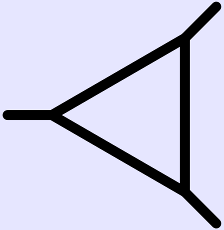
$$= -\frac{1}{\epsilon} + \log\left(m^2\right) - \frac{1}{12}\epsilon\left(6\log^2(m^2) + \pi^2\right) + \frac{1}{12}\epsilon^2\left(2\log^3(m^2) + \pi^2\log\left(m^2\right) + 4\zeta(3)\right)$$
$$+ \frac{1}{480}\epsilon^3\left(-160\zeta(3)\log\left(m^2\right) - 20\log^4(m^2) - 20\pi^2\log^2(m^2) - 3\pi^4\right) + O\left(\epsilon^4\right),$$



$$= -\frac{2}{\epsilon}\frac{e^{\gamma_E \epsilon}\,\Gamma\left(1-\epsilon\right)^2\Gamma\left(1+\epsilon\right)}{\Gamma\left(1-2\epsilon\right)}(-p^2)^{-1-\epsilon}$$

$$= -\frac{1}{\epsilon} + \log\left(-p^2\right) - \frac{1}{12}\epsilon\left(6\log^2(-p^2) - \pi^2\right) + \frac{1}{12}\epsilon^2\left(2\log^3(-p^2) - \pi^2\log\left(-p^2\right) + 28\zeta(3)\right.$$
$$\left. + \frac{1}{1440}\epsilon^3\left(-3360\zeta(3)\log\left(-p^2\right) - 60\log^4(-p^2) + 60\pi^2\log^2(-p^2) + 47\pi^4\right) + O\left(\epsilon^4\right).\right.$$

One-loop integrals

Physics $\sim$  $\sim \log, \mathrm{Li}_2, \mathrm{Li}_{m_1 \ldots m_k}$

Multiple polylogarithms

All one-loop integrals can be expressed in terms of multiple polylogarithms (MPLs). The sum representation with weight $n_1 + \ldots + n_k$ is

$$\mathrm{Li}_{n_1 \ldots n_k}(z_1, \ldots, z_k) = \sum_{m_1 > \ldots > m_k}^{\infty} \frac{z_1^{m_1}}{m_1^{n_1}} \cdots \frac{z_k^{m_k}}{m_k^{n_k}}.$$

Some familiar special cases are

$$\begin{aligned} \text{the ordinary logarithm} \quad & \mathrm{Li}_1(z) = -\log(1-z), \\ \text{classical polylogarithms} \quad & \mathrm{Li}_n(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^n}. \end{aligned}$$

Coaction

If we let $\mathcal{A}$ denote the $\mathbb{Q}$-vector space of MPLs, then $\mathcal{H} = \mathcal{A}/(\mathrm{i}\pi\mathcal{A})$ can be endowed with a coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ which is a coassociative homomorphism.
Explicitly, the coproduct acts as follows for

$$\begin{aligned} \text{the ordinary logarithm} \quad & \Delta\left(\log(z)\right) = 1 \otimes \log(z) + \log(z) \otimes 1, \\ \text{classical polylogarithms} \quad & \Delta\left(\mathrm{Li}_n(z)\right) = 1 \otimes \mathrm{Li}_n(z) + \sum_{k=0}^{n-1} \frac{1}{k!} \mathrm{Li}_{n-k}(z) \otimes \log^k(z). \end{aligned}$$

Examples of coproducts

$$\Delta\left(\mathrm{Li}_2(z)\right) = 1 \otimes \mathrm{Li}_2(z) + \mathrm{Li}_2(z) \otimes 1 - \log(1-z) \otimes \log(z)$$

$$\Delta\left(\mathrm{Li}_3(z)\right) = 1 \otimes \mathrm{Li}_3(z) + \mathrm{Li}_3(z) \otimes 1 + \mathrm{Li}_2(z) \otimes \log(z) - \frac{1}{2} \log(1-z) \otimes \log^2(z)$$

Symbol map

Coassociativity $\Rightarrow \Delta$ can be uniquely iterated.

The maximal iteration of the coproduct is called the symbol $\mathcal{S}$:

$$\mathcal{S} : \mathcal{H}_n \to \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_1$$

This means:

$\mathcal{S}$(MPL of weight n) = Linear combination of tensor products of n MPLs of weight one.

The symbol map

To find the symbol of an MPL of weight n:

1. Iteratively apply the coproduct to the MPL $n-1$ times.
2. Extract the terms in which all entries have weight one (i.e. the ordinary logarithms).

Symbol map

Some simple examples:

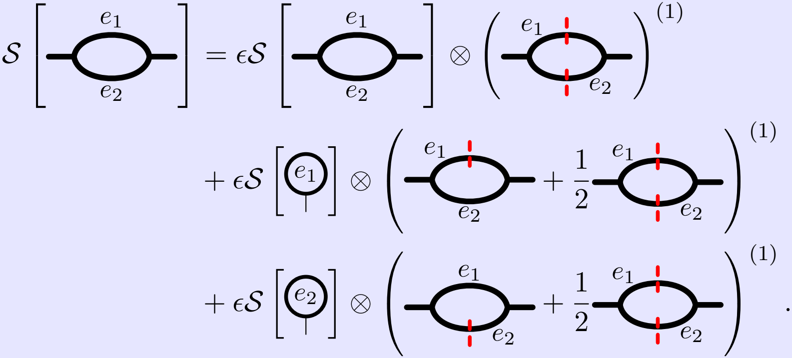
$$\Delta\left(\mathrm{Li}_2(z)\right) = 1 \otimes \mathrm{Li}_2(z) + \mathrm{Li}_2(z) \otimes 1 - \log(1-z) \otimes \log z$$

$$\Rightarrow \mathcal{S}\left(\mathrm{Li}_2(z)\right) = -\log(1-z) \otimes \log z$$

$$\begin{aligned}(\mathrm{id}\otimes\Delta)\Delta\left(\mathrm{Li}_3(z)\right) &= 1 \otimes 1 \otimes \mathrm{Li}_3(z) + 1 \otimes \mathrm{Li}_3(z) \otimes 1 \\ &+ 1 \otimes \mathrm{Li}_2(z) \otimes \log(z) + \mathrm{Li}_2(z) \otimes 1 \otimes \log(z) + \mathrm{Li}_2(z) \otimes \log(z) \otimes 1 \\ &- \frac{1}{2} 1 \otimes \log(1-z) \otimes \log^2 z - \frac{1}{2}\log(1-z) \otimes 1 \otimes \log^2 z \\ &- \frac{1}{2}\log(1-z) \otimes \log^2 z \otimes 1 - \log(1-z) \otimes \log z \otimes \log z\end{aligned}$$

$$\Rightarrow \mathcal{S}\left(\mathrm{Li}_3(z)\right) = -\log(1-z) \otimes \log z \otimes \log z$$

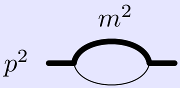
In [Abr+17], a recursive formula for the symbol entries, which relates words with n letters to words with $n+1$ letters is presented. For example, the bubble has



If we know the cut integrals and pinched symbols, we have all of the information!

Symbol recursion

Examining the recursion allows us to predict the alphabet and dictionary. Let us look at the bubble with one massive propagator whose symbol has the three-letter words

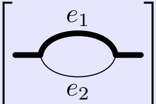


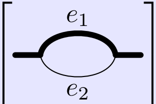
1st letter	2nd letter	3rd letter
m^2	m^2	m^2
m^2	m^2	p^2
m^2	m^2	(m^2-p^2)
m^2	p^2	p^2
m^2	p^2	(m^2-p^2)
m^2	(m^2-p^2)	p^2
m^2	(m^2-p^2)	(m^2-p^2)
(m^2-p^2)	p^2	p^2
(m^2-p^2)	p^2	(m^2-p^2)
(m^2-p^2)	(m^2-p^2)	p^2
(m^2-p^2)	(m^2-p^2)	(m^2-p^2).

Why ?

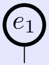
Symbol recursion

(Work done under Prof. Britto with Eliza Somerville, Mikey Whelan.) For the bubble with $m_2^2 = 0$, the recursion reads

$S\Big[$ 

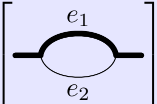
$\Big] = \epsilon S \Big[$ 

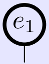
$\Big] \left(\otimes p^2 - \otimes (m^2 - p^2)^2\right)$

$+ \epsilon S \Big[$ 

$\Big] \left(\frac{1}{2} \otimes m^2 - \frac{1}{2} \otimes p^2\right).$

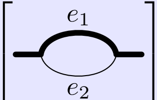
The base of the recursion involves the coefficients of ϵ^{-1} in the respective Laurent series of

$S\Big[$ 

$\Big]^{(-1)} = -\frac{1}{2}, \qquad S\Big[$ 

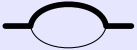
$\Big]^{(-1)} = -1$

such that the order ϵ^0 symbol words of the bubble with one massive propagator are

$S\Big[$ 

$\Big]^{(-1)} = \otimes (m^2 - p^2) - \frac{1}{2} \otimes m^2 + \frac{1}{2} \otimes p^2 - \frac{1}{2} \otimes p^2.$

Symbol recursion

Diagram	Alphabet	Dictionary
	$\{m^2, p^2, m^2 - p^2\}$	▶ Only the letter m^2 can precede m^2. ▶ The letter p^2 cannot come first.

Thank you!

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Questions

One-mass bubble letters:

$$A_1 = m_1^2, \qquad A_2 = m_2^2, \qquad A_3 = \frac{p^2}{\lambda(p^2, m_1^2, m_2^2)},$$

$$A_4 = \frac{-m_1^2 - m_2^2 - p^2 - \sqrt{\lambda(p^2, m_1^2, m_2^2)}}{-m_1^2 - m_2^2 - p^2 + \sqrt{\lambda(p^2, m_1^2, m_2^2)}}, \qquad A_5 = \frac{-m_1^2 - m_2^2 + p^2 - \sqrt{\lambda(p^2, m_1^2, m_2^2)}}{-m_1^2 - m_2^2 + p^2 + \sqrt{\lambda(p^2, m_1^2, m_2^2)}},$$