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The symbol alphabet of one-loop Feynman integrals

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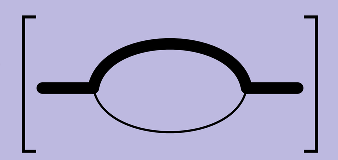
Motivation

Theoretical corrections to experimentally measurable quantities involve Feynman diagrams with loops which correspond to virtual particles in quantum field theory [1]. These loop diagrams thus form a central focus of modern particle physics but are associated with Feynman integrals which are difficult, or in some cases presently impossible, to compute [2]. Departing from physics for a moment, a clear understanding of these diagrams' analytical and algebraic nature should provide us with useful insights into bypassing sometimes intractable calculations.

Particle physics $\longleftarrow$ Theoretical corrections $\longleftarrow$ Feynman integrals

Given a Feynman diagram, there exists a map called the *symbol* [3] which produces, for example,



$$S\left[\;\cdot\;\right] \sim \left(m^2 \otimes m^2 \otimes p^2 - 2\, m^2 \otimes (m^2 - p^2) \otimes p^2\right) \epsilon^2 + \dots\,.$$

We refer to these strings of tensor products as *words* which have *letters* being functions of kinematic variables (here the mass and momentum). The output of the symbol is useful because it reveals the analytic structure of the Feynman integral without needing to compute the latter explicitly. The goal of this project was to explain the symbol alphabet (and dictionary) of various one-loop Feynman diagrams by answering the questions:

- What letters are possible? In what entry?
- Given this alphabet, what words appear?

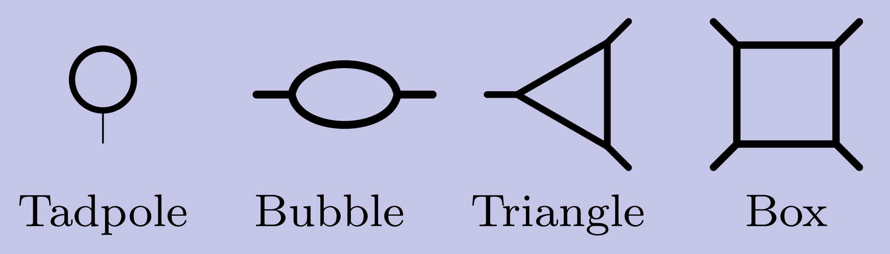
Feynman integrals

One-loop integrals

All one-loop Feynman integrals can be expressed in terms of scalar n-point integrals

$$I_n^D\left(\{p_i \cdot p_j\}; \{m_i^2\}\right) = e^{\gamma_E \epsilon} \int \frac{d^D k}{\mathrm{i}\pi^{D/2}} \prod_{j=1}^{n} \frac{1}{(k - q_j)^2 - m_j^2 + \mathrm{i}0},$$

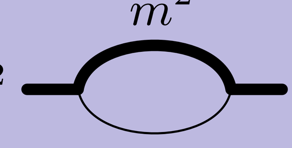
where the external momenta satisfy $\sum_i p_i = 0$ and q_i are chosen such that momentum is conserved at each vertex. In dimensional regularisation with complex variable ϵ, if $D_n \equiv 2\lceil \frac{n}{2} \rceil - 2\epsilon$ the basis of master integrals becomes $\widetilde{J}_n = I_n^{D_n}$ [1].



All one-loop Feynman integrals are expressible as multiple polylogarithms (MPLs)

$$\mathrm{Li}_{m_1 \dots m_k}(x_1, \dots, x_k) = \sum_{n_1 > \dots > n_k}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \cdots \frac{x_k^{n_k}}{n_k^{m_k}}.$$

In particular, the integrals $\widetilde{J}_n$ are expressible in terms of pure MPLs with uniform weight at each order in ϵ if we normalise them by their maximal cut: $J_n \equiv \widetilde{J}_n / j_n$.



$$p^2 \;[\text{bubble}, m^2]\; = \widetilde{J}_2\left(p^2; m^2, 0\right), \qquad \left(j_2 = \frac{2}{m^2 - p^2}\right),$$

$$\begin{aligned} J_2\left(p^2; m^2, 0\right) &= -\frac{1}{2\epsilon} e^{\gamma_E \epsilon}\, \Gamma(\epsilon + 1)\, (m^2 - p^2)^{-\epsilon}\, {}_2F_1\left(-\epsilon, 1 + \epsilon; 1 - \epsilon; \frac{p^2}{p^2 - m^2}\right) \\ &= \left(-\frac{1}{2}\right) \epsilon^{-1} + \left(\log(m^2 - p^2) - \frac{1}{2}\log(m^2)\right) \epsilon^0 \\ &\quad + \left(\mathrm{Li}_2\left(\frac{p^2}{p^2 - m^2}\right) + \frac{1}{4}\log^2(m^2) - \frac{1}{2}\log^2(m^2 - p^2)\right) \epsilon^1 + \dots\,. \end{aligned}$$

Cut integrals

The simplest version of a cut integral $\mathcal{C}_C \widetilde{J}_n$ consists of choosing a subset C of propagators and, in the integrand of $\widetilde{J}_n$, putting each propagator on-shell via a Dirac delta function. A more general definition of one-loop cut integrals $\mathcal{C}_C \widetilde{J}_n$ [4] involves the Gram determinant and the modified Cayley determinant

$$\begin{aligned} \mathrm{Gram}_C &= \det\left((q_i - q_*) \cdot (q_j - q_*)\right)_{i,j \in C \setminus *}, \\ Y_C &= \det\left(m_i^2 + m_j^2 - (q_i - q_j)^2\right)_{i,j \in C} 2^{-n}. \end{aligned}$$

For example, in the case when no determinant vanishes $\mathcal{C}_{[n]} \widetilde{J}_n \sim j_n \left(\frac{Y_{[n]}}{\mathrm{Gram}_{[n]}}\right)^{-\epsilon}$, where the maximal cut in integer dimensions is

$$j_n \equiv \lim_{\epsilon \to 0} \mathcal{C}_{[n]} \widetilde{J}_n = \begin{cases} 2^{(1-n)/2} \mathrm{i}^{(n-1)/2}\, \mathrm{Gram}_{[n]}^{-1/2} & n \text{ odd}, \\ 2^{1-n/2} \mathrm{i}^{n/2} Y_{[n]}^{-1/2} & n \text{ even}. \end{cases}$$

References

[1] Stefan Weinzierl. 2022. arXiv: 2201.03593 [hep-th].
[2] Henriette Elvang, Yu-tin Huang. 2014. arXiv: 1308.1697 [hep-th].
[3] Samuel Abreu et al. 2017. arXiv: 1704.07931v2 [hep-th].
[4] Samuel Abreu et al. 2017. arXiv: 1702.03163 [hep-th].
[5] A. B. Goncharov et al. 2010. arXiv: 1006.5703 [hep-th].
[6] Nima Arkani-Hamed, Ellis Ye Yuan. 2017. arXiv: 1712.09991 [hep-th].

The symbol map

Coaction on MPLs

With $\mathcal{A}$ the vector space of MPLs, and $\mathcal{H} = \mathcal{A}/(i\pi\mathcal{A})$, the Hopf algebra of MPLs leads to a coproduct $\Delta_{\mathrm{MPL}} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{H}$. Acting on the classic polylogarithm,

$$\Delta_{\mathrm{MPL}}(\mathrm{Li}_k(z)) = 1 \otimes \mathrm{Li}_k(z) + \sum_{n=0}^{k-1} \frac{1}{n!} \mathrm{Li}_{k-n}(z) \otimes \log^n(z).$$

Symbols

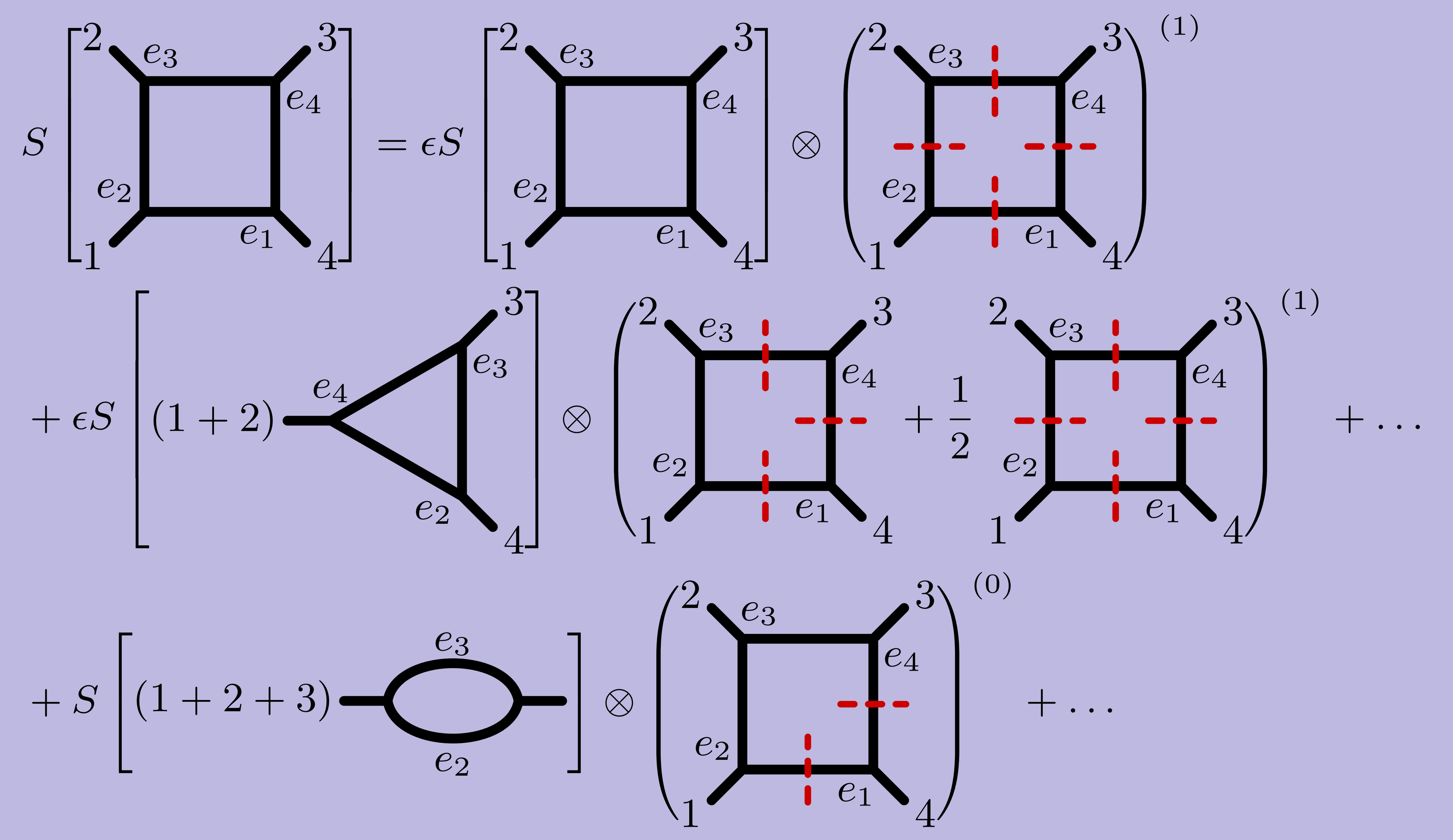
For an iterated integral expressed in *dlog-forms* [5], the *symbol* returns

$$I = \int_a^b d\log R_1 \circ \dots \circ d\log R_k \longrightarrow S[I] = R_1 \otimes \dots \otimes R_k$$

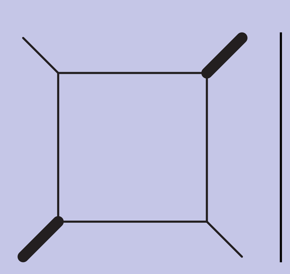
and up to constants is equivalent to the maximally iterated coaction. Since 1-loop diagrams are expressed as MPLs of uniform weight (for a given ϵ order), we can meaningfully apply the symbol. For instance, $S[\mathrm{Li}_3(z)] = -(1 - z) \otimes z \otimes z$ whence $\{z, 1 - z\}$ are letters.

Using the recursion

We computed the symbol alphabet *in full* using the symbol recursion formula [3] and data of all the relevant cuts for select diagrams. For each diagram or cut, we used the analytic solution to take an expansion in ϵ. The symbol has words with length corresponding to the order in ϵ of the term. For example, the symbol recursion for the box is



The *two-mass easy box* is the box diagram where $p_1^2, p_3^2 \neq 0$ and has the alphabet



$$\Big\{p_1^2,\, p_3^2,\, s,\, t,\; p_1^2 - s,\; p_1^2 - t,\; p_3^2 - s,\; p_3^2 - t,\; st - p_1^2 p_3^2,\quad s + t - p_1^2 - p_3^2\Big\}$$

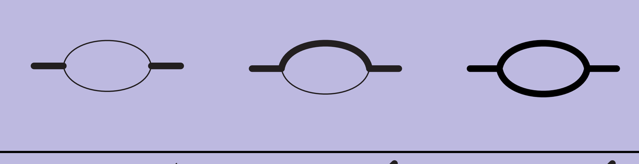
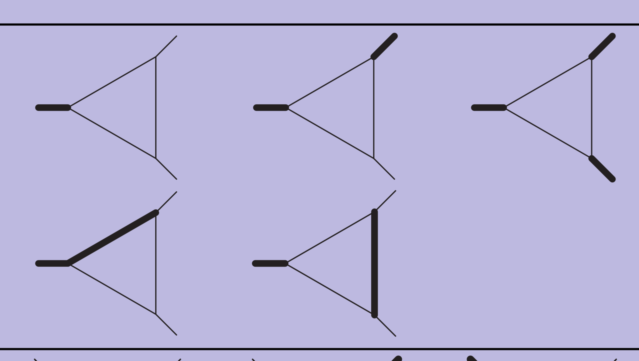
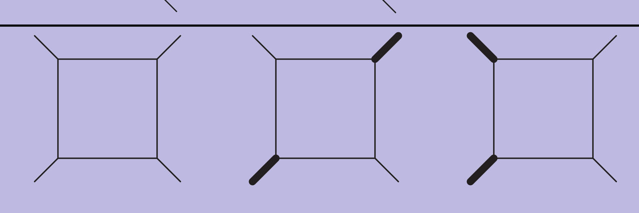
From the recursion, we determine a set of rules dictating which set of letters can sequentially occur after another set of letters. A valid word might look like

$$s \otimes s \otimes \dots \otimes s \;\otimes\; (p_1^2 - s) \;\otimes\; (st - p_1^2 p_3^2) \otimes \dots \otimes (s + t - p_1^2 - p_3^2).$$

We found that the letter $(s + t - p_1^2 - p_3^2)$ cancels at the first steps of the recursion such that it only appears from the third entry onwards. This type of cancellation is not immediately obvious from the recursion but should be explainable when looking at the formula in terms of determinants.

Conclusion

We successfully determined the symbol alphabet for some non-generic cases.

Diagram	Alphabet	Dictionary
	✓ ✓ ✓	✓ ✓ ✓
	✓ ✓ ⚙ ⚙ ⚙	✓ ✓ ✓ ✓ ✓
	✓ ✓ ⚙	✓ ✓ ⚙

Constraints on letters seem to come from relations between determinants specific to the diagram, which ultimately detail the geometry of polytopes associated with the kinematics. Linear relations [3, 4] between cut integrals may also provide an avenue to study the implications of this geometry regarding the occurrence of symbol letters. Interestingly, our method relied on cuts $\mathcal{C}_{[n]}, \mathcal{C}_{[n-1]}, \mathcal{C}_{[n-2]}$ whereas [6] claims to exhaust the symbol alphabet with cuts of two propagators $\mathcal{C}_{\{i,j\}}$. Reconciling this, and exploring cuts at orders $\epsilon^{0,1}$ in terms of determinants, should provide a clearer perspective of symbol alphabets for one-loop Feynman integrals.