A review of

Foundations of the $\mathrm{AdS}_5 \times S^5$ Superstring

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Why $\mathrm{AdS}_5 \times S^5$?

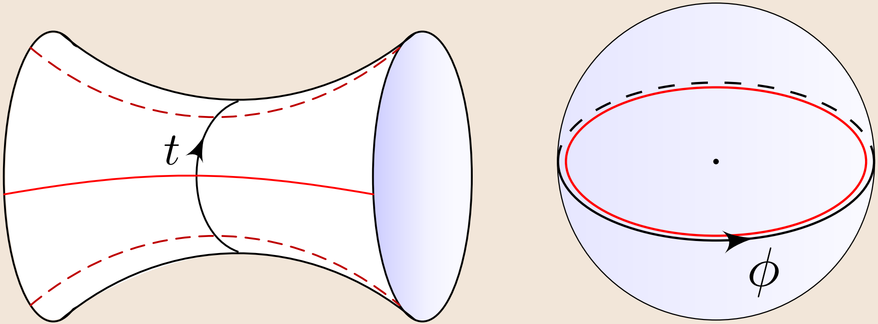
- Modern TP: *Can we reconcile Standard Model and General Relativity?*
- **String theory** seems to reproduce both simultaneously.
- Starting in 1960's, string theory produced bosons in $D = 26$.
- In 1980's, super(symmetric)string theory also produced fermions in $D = 10$.
- In 1990's, J. Maldacena conjectured **AdS/CFT correspondence**.

Anti-de Sitter spacetime	$\leftrightarrow$	**Conformal field theory**
Type IIB AdS_5 superstring		$\mathcal{N} = 4$ Super-Yang-Mills

- In this talk, review [AF09] approach to quantising $\mathrm{AdS}_5 \times S^5$ superstring.

$$\mathscr{L} \longrightarrow \mathcal{H} \longrightarrow \hat{\mathcal{H}} \longrightarrow \ldots$$

Why $\mathrm{AdS}_5 \times S^5$?



$\mathbf{x} \cdot \mathbf{x} = -R^2$ in $\mathbb{R}^{n-1,2}$

$\mathbf{x} \cdot \mathbf{x} = R^2$ in $\mathbb{R}^{n+1}$

Figure 1. Classical strings on hypersurfaces AdS_n and S^n for $n = 2$ [Tse11, Sok16].

Bosonic string theory

- Generalise relativistic point particle to one-dimensional string.

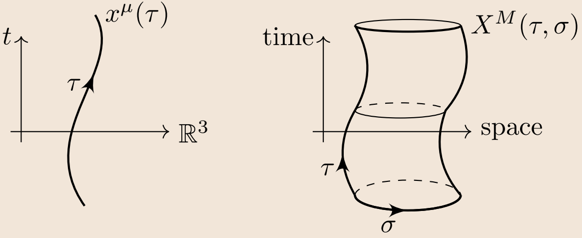


Figure 2. Worldline of particle and worldsheet of a closed string.

- Parameterise spacetime coordinates $X^M(\boldsymbol{\sigma})$ where $\boldsymbol{\sigma} = (\tau, \sigma)$, $M = 1, \ldots, 26$ and

$$\partial_\tau X^M = \dot{X}^M, \qquad \partial_\sigma X^M = X'^M.$$

- $G_{MN} \sim$ spacetime metric and $\gamma_{\alpha\beta} \sim$ metric on the **worldsheet** for $\alpha, \beta \in \{\tau, \sigma\}$.
- For closed strings, worldsheet parameterised by $\tau \in \mathbb{R}$ and $-\pi r < \sigma < \pi r$.

Bosonic string theory

- **Polyakov action** for bosonic strings [GSW88]:

$$S = \int d\tau \int d\sigma \mathscr{L} = -\frac{T}{2} \int d\tau d\sigma \; \gamma^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N G_{MN}.$$

- Solving equations of motion $\delta S / \delta \gamma_{\alpha\beta} = 0$ yields **Virasoro constraints**

$$C_1 = p_M X'^M = 0, \qquad C_2 = p_M p^M + T^2 X'_M X'^M = 0.$$

- Rewrite Polyakov action in **first-order form** as

$$\begin{aligned} S &= \int d\tau d\sigma \; \left(p_M \dot{X}^M + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}} C_1 + \frac{1}{2T\gamma^{\tau\tau}} C_2 \right) \\ &= \int d\tau d\sigma \, p_M \dot{X}^M . \end{aligned}$$

Superstring theory

Bosonic/fermionic encoded in **superalgebra** structure [Kac77].

- $\mathscr{G} = \mathscr{G}^{(\mathbf{0})} \oplus \mathscr{G}^{(\mathbf{1})}$ is $\mathbb{Z}_2$-graded such that

$$[\mathscr{G}^{(\mathbf{a})}, \mathscr{G}^{(\mathbf{b})}] \subseteq \mathscr{G}^{(\mathbf{a}+\mathbf{b})} \quad \text{mod } \mathbb{Z}_2 = \{\mathbf{0}, \mathbf{1}\}.$$

- If $M \in \mathscr{G} \equiv \mathfrak{su}(2,2|4)$, then $MH + HM^\dagger = 0$ and $\operatorname{str}\big(M\big) = \operatorname{tr}(m) - \operatorname{tr}(n) = 0$ where

even odd

$$M = \begin{pmatrix} m & \theta \\ \eta & n \end{pmatrix}, \qquad H = \begin{pmatrix} \gamma^5 & 0 \\ 0 & \mathbb{1}_4 \end{pmatrix}.$$

- Refine to $\mathbb{Z}_4$**-grading** with decomposition

$$\mathscr{G} = \mathscr{G}^{(0)} \oplus \mathscr{G}^{(1)} \oplus \mathscr{G}^{(2)} \oplus \mathscr{G}^{(3)},$$
$$M = M^{(0)} + M^{(1)} + M^{(2)} + M^{(3)},$$

according to automorphism $\Omega : \mathscr{G} \to \mathscr{G}$ where $\Omega^4(M) = M$ and

$$M^{(k)} = \frac{1}{4}\left[M + \mathrm{i}^{3k}\Omega(M) + \mathrm{i}^{2k}\Omega^2(M) + \mathrm{i}^{k}\Omega^3(M)\right].$$

Superstring theory

- Parameterise $\mathrm{AdS}_5 = \{t, z^i\}$ and $S^5 = \{\phi, y^i\}$ for $i = 1, \ldots, 4$.
- Collect $X^M \in \{t, \phi, x^\mu\}$ where $x^\mu \sim$ transversal degrees of freedom for $\mu = 1, \ldots, 8$.
- Define current $A_\alpha = -\mathfrak{g}^{-1}\partial_\alpha \mathfrak{g} \in \mathscr{G}$ where $\mathfrak{g} = \mathfrak{g}_\mathfrak{b}(t, \phi, x^\mu)\mathfrak{g}_\mathfrak{f}(\chi) \in G = \exp \mathscr{G}$.

Green-Schwarz action for $\mathrm{AdS}_5 \times S^5$ superstring

$$S = -\frac{T}{2}\int d^2\sigma \left[\gamma^{\alpha\beta}\mathrm{str}\left(A_\alpha^{(2)} A_\beta^{(2)}\right) + \kappa\varepsilon^{\alpha\beta}\mathrm{str}\left(A_\alpha^{(1)} A_\beta^{(3)}\right)\right]$$

- Introduce auxiliary $\pi = \pi^{(2)} \in \mathscr{G}^{(2)}$, the first-order form is

$$S = \int d^2\sigma \left[-\mathrm{str}\left(\pi A_\tau^{(2)}\right) + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}} C_1 + \frac{1}{2T\gamma^{\tau\tau}} C_2\right] - \frac{T}{2}\int d^2\sigma \kappa \varepsilon^{\alpha\beta} A_\alpha^{(1)} A_\beta^{(3)}$$

where this time

$$C_1 = -\mathrm{str}\left(\pi A_\sigma^{(2)}\right) = 0, \qquad C_2 = \mathrm{str}\left(\pi^2 + T^2 A_\sigma^{(2)} A_\sigma^{(2)}\right) = 0.$$

Gauge fixing

	κ**-symmetry gauge (s)**	**Light cone gauge (s, b)**
Gauge freedom	$\delta\mathscr{L} = 0$ under $\mathfrak{g} \to \mathfrak{g}e^{\epsilon(\tau,\sigma)}$ provided $\kappa = \pm 1$	$\delta\mathscr{L} = 0$ under $(\tau,\sigma) \to (\tilde{\tau},\tilde{\sigma})$ diffeomorphisms
Gauge fixing	$\chi \longrightarrow \left(\begin{array}{cc\|cc} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ \hline 0 & b^\dagger & 0 & 0 \\ -a^\dagger & 0 & 0 & 0 \end{array}\right)$	$x_+ = \frac{1}{2}(\phi + t) \to \tau,$ $p_+ = \frac{1}{2}(p_\phi - p_t) \to 1.$

Gauge fixing

- Recall for bosonic string, first-order form of action is

$$S = \int d\tau d\sigma \, p_M \dot{X}^M = \int d\tau d\sigma \, \left(p_\mu \dot{x}^\mu + p_t \dot{t} + p_\phi \dot{\phi} \right)$$
$$= \int d^2\sigma \, \left(p_\mu \dot{x}^\mu + p_- \dot{x}_+ + p_+ \dot{x}_- \right).$$

  In light cone gauge, where $x_+ = \tau$ and $p_+ = 1$,

$$S = \int d^2\sigma \, \left(p_\mu \dot{x}^\mu + p_- + \cancel{\dot{x}_-} \right) = \int d^2\sigma \, \left(p_\mu \dot{x}^\mu - \mathcal{H} \right).$$

- Hamiltonian of classical bosonic string in light cone gauge is

$$\mathcal{H} = -p_-(p_\mu, x^\mu, x'^\mu).$$

- What is p_- for superstring?

Perturbative quantisation

- For superstring, one finds

$$\begin{aligned}\mathcal{H} = \ldots = &\frac{\mathrm{i}}{2}\mathrm{str}\big(\pi\Sigma_+\mathfrak{g}(x^\mu)(\mathbb{1}+2\chi^2)\mathfrak{g}(x^\mu)\big)\\ &+\kappa\frac{T}{2}(G_+^2-G_-^2)\mathrm{str}\big(\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2}\mathcal{K}F_\sigma^{st}\mathcal{K}^{-1}\big)\\ &-\kappa\frac{T}{2}G_\mu G_\nu\mathrm{str}\big(\Sigma_\nu\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2}\Sigma_\mu\mathcal{K}F_\sigma^{st}\mathcal{K}^{-1}\big).\end{aligned}$$

3 weeks

- Not nice! Trick: rescale $\sigma \to \sigma T$ such that $2\pi r \to 2\pi r T$ and also rescale

$$x^\mu \to x^\mu/\sqrt{T}, \qquad p_\mu \to p_\mu/\sqrt{T}, \qquad \chi \to \chi/\sqrt{T}.$$

- End up with perturbative expansion in large tension limit

$$S = \int d^2\sigma \left(\mathscr{L}_2 + \frac{1}{T}\mathscr{L}_4 + \frac{1}{T^2}\mathscr{L}_6 + \mathcal{O}(T^{-3})\right) \approx \int d^2\sigma\, \mathscr{L}_2\,.$$

Perturbative quantisation

- Part of Lagrangian quadratic in original fields (x^μ, p_μ, χ) is

$$\mathscr{L}_2 = p_\mu \dot{x}^\mu - \frac{\mathrm{i}}{2}\mathrm{str}\left(\Sigma_+ \chi \dot{\chi}\right) - \mathcal{H}_2 \,,$$

where we read off

$$\mathcal{H}_2 = \frac{1}{4}P_{a\dot{a}}P^{a\dot{a}} + Y_{a\dot{a}}Y^{a\dot{a}} + Y'_{a\dot{a}}Y'^{a\dot{a}} + \frac{1}{4}P_{\alpha\dot{\alpha}}P^{\alpha\dot{\alpha}} + Z_{\alpha\dot{\alpha}}Z^{\alpha\dot{\alpha}} + Z'_{\alpha\dot{\alpha}}Z'^{\alpha\dot{\alpha}}$$
$$+ \eta^\dagger_{\alpha\dot{a}}\eta^{\alpha\dot{a}} + \frac{\kappa}{2}\eta'_{\alpha\dot{a}}\eta^{\alpha\dot{a}} - \frac{\kappa}{2}\eta'^\dagger_{\alpha\dot{a}}\eta^{\dagger\alpha\dot{a}} + \theta^\dagger_{a\dot{\alpha}}\theta^{a\dot{\alpha}} + \frac{\kappa}{2}\theta'_{a\dot{\alpha}}\theta^{a\dot{\alpha}} - \frac{\kappa}{2}\theta'^\dagger_{a\dot{\alpha}}\theta^{\dagger a\dot{\alpha}} .$$

8 Bosons

8 Fermions

- Can now promote fields to operators, i.e. **quantisation** of the superstring:

$$Y^{a\dot{a}}(\tau,\sigma) = \int_{\mathbb{R}} \frac{dp}{2\pi}\frac{1}{\sqrt{2E_p}}\left(e^{\mathrm{i}\sigma p}a_p^{a\dot{a}} + e^{-\mathrm{i}\sigma p}\varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}a^\dagger_{b\dot{b},p}\right),$$

$$\theta^{a\dot{\alpha}}(\tau,\sigma) = \int_{\mathbb{R}} \frac{dp}{2\pi}\frac{1}{\sqrt{2E_p}}\left(e^{\mathrm{i}\sigma p}f(p)a_p^{a\dot{\alpha}} + e^{-\mathrm{i}\sigma p}h(p)\varepsilon^{ab}\varepsilon^{\dot{\alpha}\dot{\beta}}a^\dagger_{b\dot{\beta},p}\right).$$

Perturbative quantisation

- In decompactification limit ($r \to \infty$) the worldsheet cylinder becomes a plane.

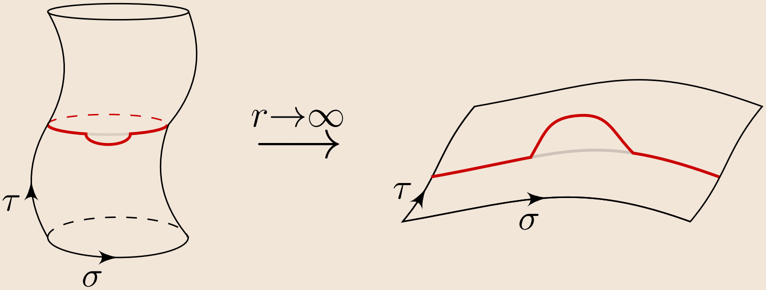


Figure 3. String excitation in decompactification limit

- Nice! Reduced problem to 1+1-dimensional QFT with 8 bosons and 8 fermions.
- What about **full** spectrum?

Conclusion

AdS/CFT

$$\mathscr{L} \longrightarrow \mathcal{H} \longrightarrow \hat{\mathcal{H}} \longrightarrow \text{Spectrum} \leftrightarrow \mathcal{N} = 4 \text{ SYM}$$

First-order formalism

κ-symmetry gauge

Light cone gauge

Thermodynamic Bethe Ansatz (TBA)

Perturbative quantisation in decompactification limit

Thank you

References

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[GSW88] M.B. Green, J.H. Schwarz, and E. Witten. *Superstring Theory: Volume 1, Introduction.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1988. ISBN: 9780521357524.

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Appendix

- Where is the spacetime hiding?

$$\mathrm{AdS}_5 \times S^5 \cong \frac{SO(4,2)}{SO(4,1)} \times \frac{SO(6)}{SO(5)} \cong \frac{SU(2,2)}{SO(4,1)} \times \frac{SU(4)}{SO(5)} \subset \frac{SU(2,2|4)}{SO(4,1) \times SO(5)}.$$

- Automorphism $\Omega(M) = -\mathcal{K} M^{st} \mathcal{K}^{-1}$ where

$$M^{st} = \begin{pmatrix} m^t & -\eta^t \\ \theta^t & n^t \end{pmatrix}, \quad \text{and} \quad \mathcal{K} = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \quad K = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

- Decomposition

$$M^{(0)} = \frac{1}{2} \begin{pmatrix} m - K m^t K^{-1} & 0 \\ 0 & n - K n^t K^{-1} \end{pmatrix}, \quad M^{(1)} = \frac{1}{2} \begin{pmatrix} 0 & \theta - \mathrm{i} K \eta^t K^{-1} \\ \theta + \mathrm{i} K \eta^t K^{-1} & 0 \end{pmatrix},$$

$$M^{(2)} = \frac{1}{2} \begin{pmatrix} m + K m^t K^{-1} & 0 \\ 0 & n + K n^t K^{-1} \end{pmatrix}, \quad M^{(3)} = \frac{1}{2} \begin{pmatrix} 0 & \theta + \mathrm{i} K \eta^t K^{-1} \\ \theta - \mathrm{i} K \eta^t K^{-1} & 0 \end{pmatrix}.$$

Appendix

- Embedding coordinates. Set $\mathfrak{g}_\mathfrak{b}(t,\phi,x^\mu) = \Lambda(t,\phi)\mathfrak{g}(x^\mu)$ where

$$\Lambda(t,\phi) = \exp\frac{\mathrm{i}}{2}\begin{pmatrix} t\gamma^5 & 0 \\ 0 & \phi\gamma^5 \end{pmatrix}, \quad \mathfrak{g}(x^\mu) = \begin{pmatrix} \frac{1}{\sqrt{1-\mathbf{z}^2/4}}[\mathbb{1}_4 + \frac{1}{2}z^i\gamma^i] & 0 \\ 0 & \frac{1}{\sqrt{1+\mathbf{y}^2/4}}[\mathbb{1}_4 + \frac{\mathrm{i}}{2}y^i\gamma^i] \end{pmatrix},$$

$$\mathfrak{g}_\mathfrak{f}(\chi) = \chi + \sqrt{\mathbb{1}_8 + \chi^2}.$$

- Light cone coordinates

$$t = x_+ - a x_-, \qquad x_+ = a\phi + (1-a)t,$$
$$\phi = x_+ + (1-a)x_-, \qquad x_- = \phi - t.$$

- Isomorphism

$$\mathfrak{su}(4) \sim \mathfrak{so}(6) = \mathrm{span}_\mathbb{R}\left\{\frac{\mathrm{i}}{2}\gamma^i, \frac{1}{4}[\gamma^i,\gamma^j]\right\}, \qquad i,j = 1,\dots,5,$$
$$\mathfrak{su}(2,2) \sim \mathfrak{so}(4,2) = \mathrm{span}_\mathbb{R}\left\{\frac{1}{2}\gamma^i, \frac{\mathrm{i}}{2}\gamma^5, \frac{1}{4}[\gamma^i,\gamma^j], \frac{\mathrm{i}}{4}[\gamma^i,\gamma^5]\right\}, \qquad i,j = 1,\dots,4.$$