Trinity College Dublin
Coláiste na Tríonóide, Baile Átha Cliath
The University of Dublin

Quantising the $\mathrm{AdS}_5 \times S^5$ Superstring

Alexander Farren supervised by Sergey Frolov

Why $\mathrm{AdS}_5 \times S^5$?

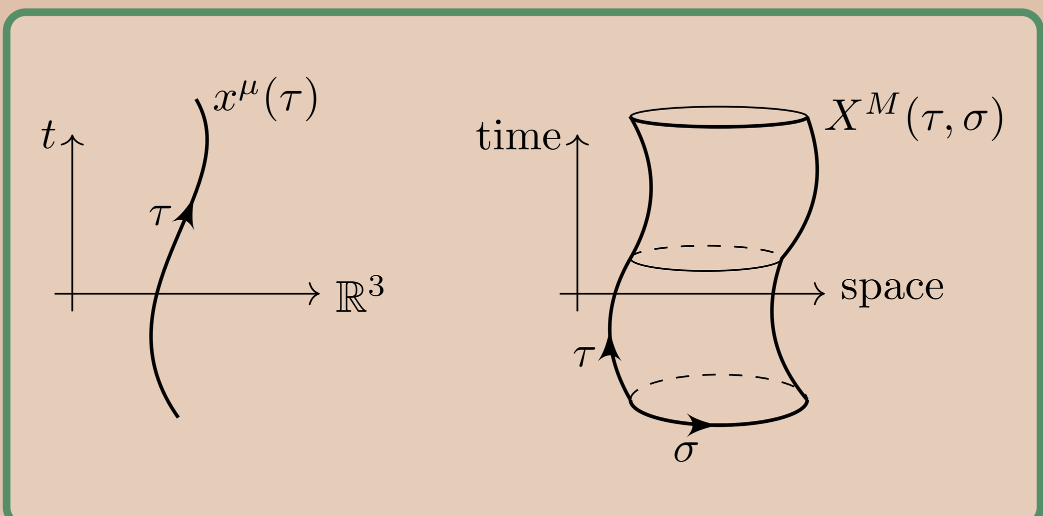
A central effort of modern theoretical physics is **quantum gravity**, a field of study dedicated to reconciling the quantum framework of particle physics with the classical picture of general relativity. Since its formulation in the 1960's, **string theory** has provided many rich insights into how these two are related. In particular, in 1997, Juan Maldacena put forward what is now known as the **AdS/CFT correspondence**:

Anti-de Sitter spacetime	$\leftrightarrow$	**Conformal field theory**
Type IIB AdS_5 superstring		$\mathcal{N} = 4$ Super Yang-Mills

One needs the **full spectrum** of operators of the quantised string in the AdS spacetime to use this duality. This project is a review of [1] by Gleb Arutyunov and Sergey Frolov, which presents a procedure for quantising the $\mathrm{AdS}_5 \times S^5$ superstring by reducing the problem to finding the spectrum of scattering states in a 2D quantum field theory.

Bosonic string theory

The basic idea is to generalise a relativistic point particle to a one-dimensional string which lives in $D = 26$ spacetime. The modes of vibration can be identified as bosons.



Worldline of particle and worldsheet of a closed string.

Now parameterise spacetime coordinates $X^M(\boldsymbol{\sigma})$ by $\boldsymbol{\sigma} = (\tau, \sigma)$, where $M = 1, ..., 26$. Spacetime geometry is desribed by G_{MN} and the worldsheet by $\gamma_{\alpha\beta}$ for $\alpha \in \{\tau, \sigma\}$. For closed strings, worldsheet parameterised by $\tau \in \mathbb{R}$ and $-\pi r < \sigma < \pi r$. Minimising the worldsheet area gives **Polyakov action** for bosonic strings [2]

$$S = \iint d\tau d\sigma\, \mathscr{L} = -\frac{T}{2} \iint d\tau d\sigma\, \gamma^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N G_{MN}.$$

Solving equations of motion $\delta S / \delta \gamma_{\alpha\beta} = 0$ yields **Virasoro constraints**

$$C_1 = p_M X'^M = 0, \qquad C_2 = p_M p^M + T^2 X'_M X'^M = 0.$$

Solving $C_1 = C_2 = 0$, can rewrite Polyakov action in **first-order form** as

$$S = \iint d\tau d\sigma \left(p_M \dot{X}^M + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}} C_1 + \frac{1}{2T\gamma^{\tau\tau}} C_2\right) = \iint d\tau d\sigma\, p_M \dot{X}^M.$$

Suppose we could change to light cone coordinates where $p_M \dot{X}^M = p_\mu \dot{x}^\mu + p_- \dot{x}_+ + p_+ \dot{x}_-$ and x^μ are the physical coordinates. In light cone gauge, where $x_+ = \tau$ and $p_+ = 1$, we see that the Hamiltonian density of a classical bosonic string in light cone gauge is $\mathcal{H} = -p_-(p_\mu, x^\mu, x'^\mu)$. We want to answer: What is p_- for a superstring?

Superstring theory

This time $D = 10$ and the bosonic/fermionic paradigm is encoded in a **superalgebra** structure [3]. A $\mathbb{Z}_2$-graded Lie algebra $\mathscr{G} = \mathscr{G}^{(\mathbf{0})} \oplus \mathscr{G}^{(\mathbf{1})}$ is equipped with the Lie bracket

$$[\mathfrak{a}, \mathfrak{b}] = -(-1)^{|\mathfrak{a}||\mathfrak{b}|}[\mathfrak{b}, \mathfrak{a}], \qquad [\mathscr{G}^{(\mathbf{a})}, \mathscr{G}^{(\mathbf{b})}] \subseteq \mathscr{G}^{(\mathbf{a}+\mathbf{b})}.$$

If $M \in \mathscr{G} \equiv \mathfrak{su}(2,2|4)$, then $MH + HM^\dagger = 0$ and $\mathrm{str}\big(M\big) = 0$ where

$$M = \begin{array}{c} \text{even} \quad \text{odd} \\ \begin{pmatrix} m & \theta \\ \eta & n \end{pmatrix} \end{array}, \qquad H = \begin{pmatrix} \gamma^5 & 0 \\ 0 & \mathbb{1}_4 \end{pmatrix}.$$

Refine to $\mathbb{Z}_4$**-grading** with decomposition $\mathscr{G} = \mathscr{G}^{(0)} \oplus \mathscr{G}^{(1)} \oplus \mathscr{G}^{(2)} \oplus \mathscr{G}^{(3)}$. In this case, parameterise $\mathrm{AdS}_5 = \{t, z^i\}$ and $S^5 = \{\phi, y^i\}$ for $i = 1, ..., 4$. Again we can collect $X^M \in \{t, \phi, x^\mu\}$ where x^μ are the transversal degrees of freedom for $\mu = 1, ..., 8$. Define the $\mathfrak{su}(2,2|4)$ one-form current which embeds $\mathrm{AdS}_5 \times S^5 \subset SU(2,2|4)/(SO(4,1) \times SO(5))$:

$$A_\alpha = -\mathfrak{g}^{-1}\partial_\alpha \mathfrak{g} \in \mathscr{G}, \qquad \mathfrak{g} = \Lambda(t,\phi)\mathfrak{g}_\mathfrak{b}(x^\mu)\mathfrak{g}_\mathfrak{f}(\chi) \in \exp \mathscr{G}.$$

Putting all these ingredients together, we can finally understand the relevant action.

$$S = -\frac{T}{2} \int d^2\sigma \left[\gamma^{\alpha\beta}\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) + \kappa\varepsilon^{\alpha\beta}\mathrm{str}\big(A^{(1)}_\alpha A^{(3)}_\beta\big)\right]$$

Green-Schwarz action for the $\mathrm{AdS}_5 \times S^5$ superstring

References

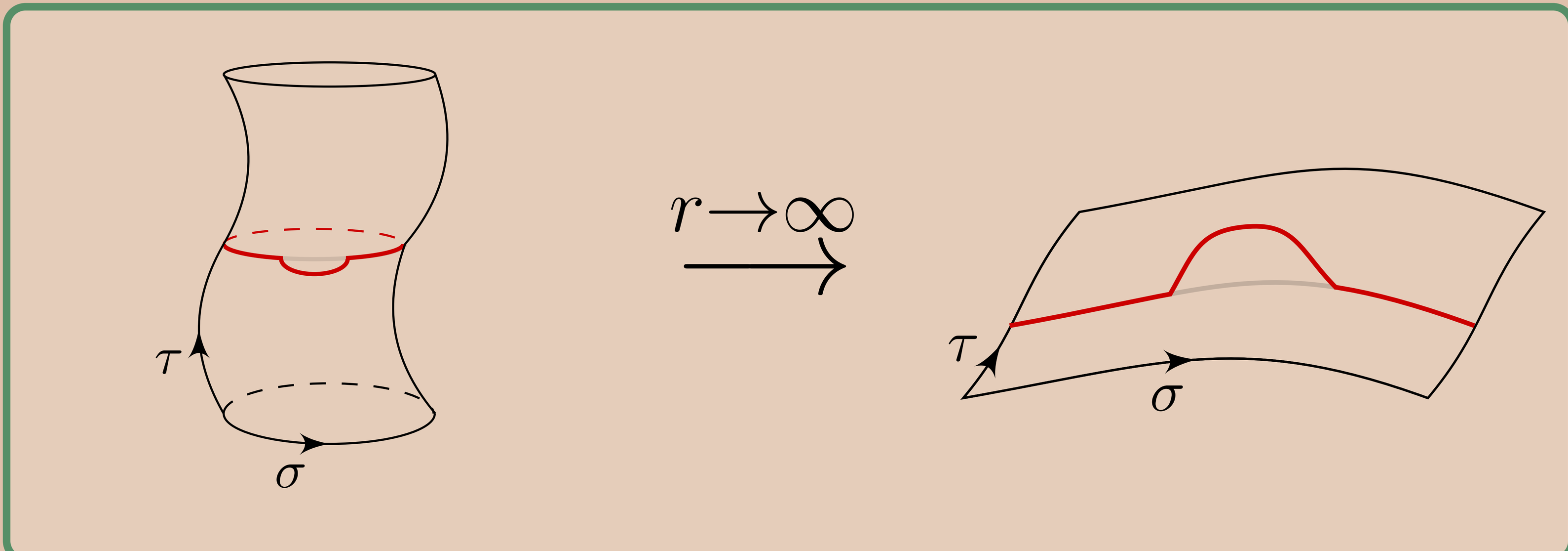
[1] Arutyunov, Frolov. *Foundations of the $\mathrm{AdS}_5 \times S^5$ Superstring. Part I.* 2009.
[2] Green, Schwarz, Witten. *Superstring Theory: Volume 1, Introduction.* 1988.
[3] Kac. *Lie superalgebras.* 1977.
[4] Frolov, Plefka, Zamaklar. *The $\mathrm{AdS}_5 \times S^5$ superstring in light-cone gauge and its Bethe equations.* 2006.

Quantising the superstring

To simplify quantisation step, i.e. $\mathcal{H} = -p_- \to \mathbb{H}$, we can fix the following gauges.

	κ-symmetry gauge (s)	**Light cone gauge (s, b)**
Freedom	$\mathfrak{g} \to \mathfrak{g}e^{\epsilon(\tau,\sigma)}$ provided $\kappa = \pm 1$	$(\tau, \sigma) \to (\tilde{\tau}, \tilde{\sigma})$
Fixing	$\chi \longrightarrow \left(\begin{array}{cc\|cc} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ \hline 0 & b^\dagger & 0 & 0 \\ -a^\dagger & 0 & 0 & 0 \end{array}\right)$	$x_+ = \frac{1}{2}(\phi + t) \to \tau$, $p_+ = \frac{1}{2}(p_\phi - p_t) \to 1$.

The resulting gauge-fixed Lagrangian gives us a classical Hamiltonian which is still too complicated/non-linear [4]. We resort to rescaling $\sigma \to \sigma T$ such that the worldsheet circumference becomes $2\pi r T$ while we inversely rescale $(x^\mu, p_\mu, \chi) \to (x^\mu, p_\mu, \chi)/\sqrt{T}$. The radius r is taken to infinity, and we consider the large tension limit $T \gg 1$.



String excitation in the decompactification limit.

The part of Lagrangian quadratic in the physical fields (x^μ, p_μ, χ) is

$$\mathscr{L}_2 = p_\mu \dot{x}^\mu - \frac{\mathrm{i}}{2}\mathrm{str}\big(\Sigma_+ \chi \dot{\chi}\big) - \mathcal{H}_2\,,$$

where we define the quadratic Hamiltonian in terms of two-index fields

$$\mathcal{H}_2 = \overbrace{\frac{1}{4}P_{a\dot{a}}P^{a\dot{a}} + Y_{a\dot{a}}Y^{a\dot{a}} + Y'_{a\dot{a}}Y'^{a\dot{a}} + \frac{1}{4}P_{\alpha\dot{\alpha}}P^{\alpha\dot{\alpha}} + Z_{\alpha\dot{\alpha}}Z^{\alpha\dot{\alpha}} + Z'_{\alpha\dot{\alpha}}Z'^{\alpha\dot{\alpha}}}^{8\ \text{Bosons}}$$
$$+ \underbrace{\eta^\dagger_{\alpha\dot{a}}\eta^{\alpha\dot{a}} + \frac{\kappa}{2}\eta'_{\alpha\dot{a}}\eta^{\alpha\dot{a}} - \frac{\kappa}{2}\eta'^\dagger_{\alpha\dot{a}}\eta^{\dagger\alpha\dot{a}} + \theta^\dagger_{a\dot{\alpha}}\theta^{a\dot{\alpha}} + \frac{\kappa}{2}\theta'_{a\dot{\alpha}}\theta^{a\dot{\alpha}} - \frac{\kappa}{2}\theta'^\dagger_{a\dot{\alpha}}\theta^{\dagger a\dot{\alpha}}}_{8\ \text{Fermions}}.$$

These indices correspond to the linear transformation rules of $\mathfrak{g} \to G \cdot \mathfrak{g}$ where

$$G \cdot \mathfrak{g} = \Lambda \cdot G\mathfrak{g}_\mathfrak{b}G^{-1} \cdot G\mathfrak{g}_\mathfrak{f}G^{-1} \cdot G, \qquad G = \mathrm{diag}\left(SU(2)_\alpha, SU(2)_{\dot{\alpha}}, SU(2)_a, SU(2)_{\dot{a}}\right)$$

where $a = 1, 2$, $\dot{a} = \dot{1}, \dot{2}$ are even while $\alpha = 3, 4$, $\dot{\alpha} = \dot{3}, \dot{4}$ are odd, i.e. $|a| = |\dot{a}| = 0$ while $|\alpha| = |\dot{\alpha}| = 1$. We can now promote fields to operators, i.e. **quantisation**:

$$Y^{a\dot{a}}(\tau,\sigma) = \int_\mathbb{R} \frac{dp}{2\pi}\frac{1}{\sqrt{2E_p}}\left(a^{a\dot{a}}(p)e^{\mathrm{i}\sigma p} + \varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}a^\dagger_{b\dot{b}}(p)e^{-\mathrm{i}\sigma p}\right),$$
$$\theta^{a\dot{\alpha}}(\tau,\sigma) = \int_\mathbb{R} \frac{dp}{2\pi}\frac{1}{\sqrt{2E_p}}\left(f(p)a^{a\dot{\alpha}}(p)e^{\mathrm{i}\sigma p} + h(p)\varepsilon^{ab}\varepsilon^{\dot{\alpha}\dot{\beta}}a^\dagger_{b\dot{\beta}}(p)e^{-\mathrm{i}\sigma p}\right).$$

One can choose $f(p)$ and $h(p)$ such that the quadratic Hamiltonian operator of the superstring is diagonal. Grouping $M = (a|\alpha)$, we get the standard harmonic oscillator.

$$\mathbb{H}_2 = \int_\mathbb{R} \frac{dp}{2\pi}\omega_p\left(a^\dagger_{M\dot{M}}(p)a^{M\dot{M}}(p)\right), \qquad \left[a^{M\dot{M}}(p), a^\dagger_{N\dot{N}}(p')\right\} = (2\pi)\delta^M_N\delta^{\dot{M}}_{\dot{N}}\delta(p - p').$$

Hamiltonian of the quantised $\mathrm{AdS}_5 \times S^5$ superstring (?)

The energy eigenstates are the Fock space spanned by Q-particle states

$$|\Psi\rangle = a^\dagger_{M_1\dot{M}_1}(p_1)a^\dagger_{M_2\dot{M}_2}(p_2)\cdots a^\dagger_{M_Q\dot{M}_Q}(p_Q)|0\rangle,$$

and the spectrum is clearly $\mathbb{H}_2|\Psi\rangle = E|\psi\rangle$ with $E = \sum_{i=1}^Q \omega_i = \sum_{i=1}^Q \sqrt{1 + p_i^2}$. In fact, the $\mathcal{S}$-matrix factorises into two-body $\mathcal{S}$-matrices (a feature of integrable models). The conserved charges of this QFT form the **symmetry algebra** with charges satisfying

$$\{\mathbb{Q}^a_\alpha, \mathbb{Q}^{\dagger\beta}_b\} = \delta^a_b \mathbb{R}_\alpha{}^\beta + \delta^\beta_\alpha \mathbb{L}_b{}^a + \frac{1}{2}\delta^a_b\delta^\beta_\alpha\mathbb{H}, \qquad \{\mathbb{Q}^a_\alpha, \mathbb{Q}^b_\beta\} = \varepsilon_{\alpha\beta}\varepsilon^{ab}\mathbb{C}, \quad \ldots$$

Conclusion

We were able to bring the Green-Schwarz Lagrangian into diagonal form (in terms of the two-index ladder operators) by perturbatively expanding in the large tension limit. But, remember that we were after the *full* spectrum of the $\mathrm{AdS}_5 \times S^5$ superstring.

$$\mathscr{L} \longrightarrow \mathcal{H} \longrightarrow \mathbb{H}_2 \longrightarrow \text{Spectrum} \leftrightarrow \mathcal{N} = 4\ \text{SYM}$$

One can 'recompactify' the plane to a cylinder using the thermodynamic Bethe ansatz (TBA) which is related to the integrability of, for example, spin-chain models. It is remarkable and surprising that, by analysing the two-body scattering of *worldsheet* excitations, one can retrieve the full spectrum of the superstring itself. The gauge-string duality for this example has been extensively studied and agreement has been shown up to 5 loops in the QFT. Where does the $\mathrm{AdS}_5 \times S^5$ superstring stand now?

"There is no proof, but we have no doubt." – Sergey