Fermion-Monopole Scattering

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Abstract

The issue of fermions scattering off monopoles has brought to light many subtleties of gauge theory. In this essay, we will study massless s-wave fermions scattering off a monopole. The solutions will be derived, and connected to non-Abelian monopoles. We will then be wise enough to reach two consequences, namely the unitarity paradox and the Callan-Rubakov effect.

Acknowledgments

Thanks to David for his guidance throughout the year. This essay is dedicated to my Granda.

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1 Introduction

What happens when you throw an electron at a magnetic monopole?

Imagine the following. It's a Friday night and you're enjoying a pleasant game of darts with mates at your favourite bar. Even if you've never thrown a dart in your life, you manage to spin the dart in just the right way and you automatically make a bullseye. But don't get too excited: when you go to pick the dart back up, it will have somehow spun around to face outward and you're at great risk of pricking your finger Although not as colourfully, this story does in fact happen when a right-handed electron in the lowest angular momentum mode approaches a monopole. It will somehow come back as:

$$e_R^- \; + \; m \; \longrightarrow \; e_L^- \; + \; m. \tag{1.1}$$

Magnetic monopoles are infamous objects in theoretical physics. Long believed to be in conflict with the laws of electromagnetism, Paul Dirac showed us in 1931 that in quantum mechanics their existence is perfectly acceptable as long as electric charge is quantised. The electric charges in the Standard Model are indeed quantised and so magnetic monopoles *can* exist in our Universe.

It wasn't until the 1970's that people realised that electrically charged fermions behaved oddly when paired with monopoles. In particular, the solutions of the Dirac equation when the vector potential is that of a monopole has lowest angular momentum modes which exhibit a unique feature: some fermions are purely incoming solutions, while others are outgoing solutions. This is summarised in Table 1.

$U(1)$	Helicity	Example	Direction
$+$	$+$	e_R^+	outgoing
$+$	$-$	e_L^+	incoming
$-$	$+$	e_R^-	incoming
$-$	$-$	e_L^-	outgoing

Table 1. The possible ingoing and outgoing s-wave fermions in the field of a Dirac monopole. For massless fermions, the helicities $+, -$ coincide with chiralities R,L.

Owing to their spherical symmetry in more conventional quantum mechanical scenarios, we call these lowest angular momentum modes *s-waves.* Much of this essay will be spent understanding their origin and why they are special.

We will reach two dramatic consequences of massless s-wave fermions scattering off magnetic monopoles:

1. The unitary paradox. We will see that for particles of the Standard Model (quarks and leptons)

$$u_1 \; + \; u_2 \; + \; m \longrightarrow \; \overline{d}_3 \; + \; e^+ \; + \; m \tag{1.2}$$

is allowed while, in an effective 2d picture,

$$u_1 \; + \; m \longrightarrow \; \overline{u}_2 \; + \overline{d}_3 \; + \; e^+ \; + \; m \tag{1.3}$$

is not, despite naive intuition from crossing symmetry. Why is this the case? It turns out there is no outgoing state which obeys all the required conservation laws of QM/QFT. This issue has been called the unitarity paradox. There is, however, an outgoing state with *fractional* fermion numbers corresponding to an ingoing u_1:

$$u_1 \; + \; m \; \longrightarrow \frac{1}{2}\left(\overline{u}_2 + \overline{d}_3 + e^+\right) \; + \; m. \tag{1.4}$$

2. The Callan-Rubakov effect. As mentioned, the process

$$u_1 \ + \ u_2 \ + \ \textcircled{m} \longrightarrow \ \overline{d}_3 \ + \ e^+ \ + \ \textcircled{m} \tag{1.5}$$

is allowed in the presence of the monopole. This can be reformulated as

$$\text{proton} \ + \ \textcircled{m} \longrightarrow \ e^+ \ + \ \textcircled{m} \ + \ \text{pions.} \tag{1.6}$$

While this process violates baryon number this is not surprising since, in the $SU(5)$ GUT we consider, the vector bosons inside the monopole core can carry baryon number away from the scattering fermions. However this is process is not merely *allowed*, it is *catalysed* in the presence of a monopole. This is because, unlike typical perturbative exchanges between gauge bosons, the cross section for this process is not suppressed by the GUT scale m_X. We will see the origins of this effect dating back to the 80's [1, 2].

The essay is organised as follows. Section 2 will be dedicated to the formulation of a fermion scattering off a monopole, and solving for solutions of the Dirac equation in this scenario. We will finish by seeing how the Dirac point-like monopole emerges from more realistic non-Abelian gauge groups. In Section 3, we will apply these lessons to uncover (i) the unitarity paradox which emerges from the peculiar dynamics of s-waves, and (ii) we will see the Callan-Rubakov effect in action. To answer some questions about both consequences, we will learn the language of bosonisation which will be used to qualitatively explain the scattering of massless s-wave fermions in an effective $(1+1)$-dimensional picture.

2 Fermion-monopole scattering: rise of the s-wave

In this section, we flesh out the two main characters of this story: magnetic monopole and s-wave fermions. We will start by studying the simplest version of a monopole: the Dirac monopole. This is a pointlike object which radiates out a magnetic field. We will see that subtleties arise when one solves the Dirac equation in the presence of such a magnetic source. For instance, we will see the angular momentum of the system must be modified. This will lead to an unusual spectrum of states for which the lowest angular momentum modes, s-waves, have left-handed and right-handed parts going exclusively in *or* out from the monopole. To conclude, we will touch on more general monopoles first uncovered by 't Hooft and Polyakov.

2.1 Dirac monopoles

In analogy with an electric charge, a magnetic monopole is a source which produces a radial magnetic field. If we consider a monopole with 'magnetic charge' g sitting at the origin, then it produces the field

$$\mathbf{B}(r) = \frac{g}{4\pi r^2}\hat{\mathbf{r}}. \tag{2.1}$$

Dirac was the first to show that these point monopoles are consistent with quantum mechanics [3], despite at first glance a contradiction with the famous law

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \;\sim\; \text{magnetic monopoles don't exist.} \tag{2.2}$$

To find a loophole in this statement, we can try to construct a vector potential $\mathbf{A}(\mathbf{x})$ which produces the magnetic field (2.1) while somehow satisfying $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$. If we guess $\mathbf{A} \sim \hat{\boldsymbol{\phi}}$, Choose $A_0 = 0$. then the definition $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ imposes the differential equations

$$\frac{1}{r}\partial_r\left(A_\phi r\right) = 0, \quad \frac{1}{r\sin\theta}\partial_\theta\left(A_\phi\sin\theta\right) = \frac{g}{4\pi r^2}, \qquad \frac{1}{r\sin\theta}\partial_\phi\left(A_\phi\right) = 0, \tag{2.3}$$

which are easily integrated to give the gauge field

$$\mathbf{A}(r,\phi,\theta) = \mathbf{A}(r,\theta) = \frac{g}{4\pi}\frac{C-\cos\theta}{r\sin\theta}\hat{\boldsymbol{\phi}} \tag{2.4}$$

for some constant of integration C. We notice that this vector potential is singular at $\theta = 0$ (the north pole) and $\theta = \pi$ (the south pole). However, if we take $C = +1$ then by expanding the trigonometric functions we see the $\theta = 0$ singularity goes away, while if we take $C = -1$ the $\theta = \pi$ singularity goes away. This is not a coincidence. If one takes the curl $\boldsymbol{\nabla}\times\mathbf{A}$, this time being careful of the potential singularities, one finds [4, 5]

$$\mathbf{B} = \frac{g}{4\pi r^2}\hat{\mathbf{r}} + \frac{1}{2}g\left[(1+C)\Theta(-z) - (1-C)\Theta(z)\right]\delta(x)\delta(y)\hat{\mathbf{z}}. \tag{2.5}$$

This is the magnetic field of a point monopole *and* an infinitely long and thin solenoid emanating from the core, along the z-axis. Clearly if $C = +1$ then this Dirac string singularity only extends along the negative z-axis, while for $C = -1$ it runs along the positive z-axis. The issue is that we tried to describe the gauge potential on $\mathbb{R}^3$ instead of $\mathbb{R}^3\backslash\{0\}$. To circumvent this singular definition, we separate the (r,ϕ,θ) space into two regions (see Figure 1) with vector potentials [6]

$$\mathbf{A}_a = \frac{g}{4\pi r}\frac{+1-\cos\theta}{\sin\theta}\hat{\boldsymbol{\phi}} \qquad \text{in region } R_a = \left\{0 \le \theta \le \frac{\pi}{2}\right\}, \tag{2.6}$$

$$\mathbf{A}_b = \frac{g}{4\pi r}\frac{-1-\cos\theta}{\sin\theta}\hat{\boldsymbol{\phi}} \qquad \text{in region } R_b = \left\{\frac{\pi}{2} \le \theta \le \pi\right\}. \tag{2.7}$$

Both potentials separately kill the string singularity (2.5) and generate the proper magnetic field (2.1) in their respective hemispheres. But, to preserve a coherent description of the physics, we must now compensate for the piecewise definition and require that the two potentials must be equal *at least* up to a gauge transformation $\Omega(r,\phi,\theta) \in U(1)$ on the overlap $R_a \cap R_b = \{\theta = \pi/2\}$, i.e.

$$(\mathbf{A}_a - \mathbf{A}_b)\Big|_{\theta=\pi/2} \sim \frac{1}{\mathrm{i}e}\Omega^{-1}\boldsymbol{\nabla}\Omega. \tag{2.8}$$

Choosing the static gauge $A_0 = 0$, the residual gauge transformations are functions of space only, $\Omega = \Omega(\mathbf{x})$. (It is perfectly sensible to assume the monopole is fixed, as we will see later on.) We need one more ingredient to proceed with the quantum mechanics: the matter content of the system. This essay will focus on the behaviour of electrically charged fermions in the vicinity of the magnetic monopole. For a particle with charge q_e in units of e, we can indeed find such an Ω satisfying

$$A_{a,\phi} - A_{b,\phi} = \frac{g}{2\pi}\frac{1}{r\sin\theta} = \frac{1}{\mathrm{i}q_e e}\Omega^{-1}\frac{1}{r\sin\theta}\partial_\phi\Omega \tag{2.9}$$

as long as we take the gauge transformation to be[1]

$$\Omega(r,\phi,\theta) = \Omega(\phi) = e^{\mathrm{i}2\kappa\phi}, \qquad \text{where} \qquad \kappa \equiv \frac{q_e e g}{4\pi}. \tag{2.10}$$

Just as we separately defined the vector potential in the upper and lower hemispheres, to avoid the Dirac string singularity we describe the particle's wavefunction using Ψ_a in region R_a and Ψ_b in region R_b. The gauge transformation (2.10) must simultaneously take Ψ_b to Ψ_a such that

$$U(1) : \Psi_b(r,\phi,\theta) \longmapsto \Psi_a(r,\phi,\theta) = \Omega(\phi)\Psi_b(r,\phi,\theta), \tag{2.11}$$

and, in particular, must be single valued in ϕ for the phase of the fermion's wavefunction Ψ to be well-defined around the equator. The charges must consequently satisfy the charge quantisation condition

$$2\kappa = \frac{q_e e g}{2\pi} \in \mathbb{Z}. \tag{2.12}$$

For example, if we take a positron with $q_e = +1$, then $eg = 2\pi n$ for some integer n. Dirac creatively derived this relation in 1931 [3], long before the details of gauge theory were understood. In the modern perspective, we view n as a winding number corresponding to a topological invariant associated with the gauge group $U(1)$ [5, 7].

To reflect this quantisation, we shall redefine $g \to q_m g$ such that a Dirac monopole has integer magnetic charge q_m in units of the smallest charge $g = 2\pi/e$, in direct analogy with particles having electric charge q_e in units of e. A *minimal* monopole has $q_m = \pm 1$.

Due to our separation of space into two regions, we can only talk about the wavefunction—as well as functions like $\mathbf{A}(\mathbf{x})$—within only one region at a time. However, we can always relate functions as

$$\Psi_b = \Psi_a e^{-\mathrm{i}2\kappa\phi} \tag{2.13}$$

For the sake of simplicity, from this moment onward we will work in region R_a unless specified, where

$$\mathbf{A}(r,\theta) = \frac{q_m g}{4\pi}\frac{1-\cos\theta}{\sin\theta}\hat{\boldsymbol{\phi}}. \tag{2.14}$$

Of course, any results we find can be bridged to R_b via (2.13).

[1] We use natural units where $\hbar = c = \varepsilon_0 = 1$.

The quantisation condition (2.12) is a rather alluring consequence of the existence of magnetic monopoles: if such an object exists, all particles would have integer q_e with the right choice of $g = 2\pi/e$. Thus, the existence of monopoles could explain why we've only ever measured charges which are an integral multiple of that of the electron. (For quarks, the monopole's *color* magnetic field must also be taken into account. We will elaborate on this later.) Let's also state the contrapositive: if there exists just one particle with an irrational electric charge, then there cannot exist magnetic monopoles. This is surely a more interesting perspective. Nowadays, we know the Standard Model is subject to stringent consistency conditions imposed by anomaly cancellations which alone enforce electric charge quantisation, without involving any monopoles. Thus—in contrast with what is taught to undergraduates—the Standard Model plainly allows for the existence of monopoles. Detecting one experimentally is another matter: so far, none have been observed [8].

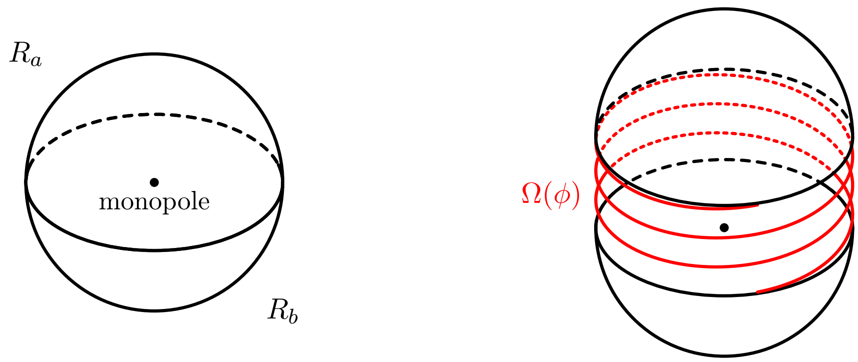

Figure 1. The separate definition of $\mathbf{A}$ in two hemispheres R_a and R_b prevents the unphysical Dirac string (2.5) from ever appearing. 'Gluing' the two hemispheres together on the intersection $R_a \cap R_b$ via a gauge transformation $\Omega(\phi)$ with winding number $n \in Z$ leads to charge quantisation $\kappa = n/2$ (2.12).

We have yet to see a most important detail of magnetic monopoles. When we work out the hydrogen problem (an electron orbiting a fixed proton) as an undergraduate, we exploit the fact that angular momentum is conserved to label the energies and stationary states by angular momentum quantum numbers j and m_j. Separation of variables then allows us to express energy eigenstates as a radial modification of the angular momentum eigenstates (spherical harmonics). In short, the conserved angular momentum of the system proved central in deriving the spectrum.

In the next section we will study the quantum mechanics of an electrically charged fermion in the presence of a fixed Dirac monopole. Inspired by the flagrant similarity between the hydrogen problem and our setup, we are strongly invited to direct our attention to the angular momentum of the system. The orbital angular momentum of particle minimally charged under $U(1)$ is typically

$$\tilde{\mathbf{L}} = \mathbf{r} \times (\mathbf{p} - q_e e \mathbf{A})\,. \tag{2.15}$$

We can ask whether the components $\tilde{L}_i$ obey the usual commutation relations. Using the fact that $\varepsilon_{kij}\varepsilon_{kab} = \delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}$, we find the action on a test function $f = f(\mathbf{x})$ to be

$$\tilde{L}_i \tilde{L}_j(f) = -x_j \partial_i f + \mathrm{i} q_e e x_j A_i f + \mathrm{i} q_e e\ \varepsilon_{ikl}\varepsilon_{jmn} x_k x_m \partial_l A_n f + \text{ something symmetric in } i \leftrightarrow j \tag{2.16}$$

such that the orbital angular momentum commutator gives

$$\begin{aligned}
[\tilde{L}_i, \tilde{L}_j](f) &= (x_i \partial_j f - x_j \partial_i f) - \mathrm{i} q_e e\,(x_i A_j - x_j A_i)\, f + \mathrm{i} q_e e \varepsilon_{ikl}\varepsilon_{jmn} x_k x_m\,(\partial_l A_n - \partial_n A_l)\, f \\
&= \mathrm{i}\varepsilon_{kij}\varepsilon_{kab}\,(x_a p_b - q_e e x_a A_b)\, f + \mathrm{i} q_e e \varepsilon_{ikl}\varepsilon_{jmn} x_k x_m \varepsilon_{cln}\varepsilon_{cab}\partial_a A_b f \\
&= \mathrm{i}\varepsilon_{ijk}\tilde{L}_k(f) + \mathrm{i}\,\frac{q_e e q_m g}{4\pi}\varepsilon_{ijk}\hat{r}_k f
\end{aligned} \tag{2.17}$$

where $\hat{r}_k = x_k/r$ and we used the definition of the gauge field (2.14), i.e. $\varepsilon_{ijk}\partial_j A_k = \hat{r}_i q_m g/4\pi r^2$. We have thus found an additional term preventing $\{\tilde{L}_i\}$ from closing to the usual $\mathfrak{su}(2)$ algebra. To retrieve the familiar properties of angular momentum, we would like to suitably modify $\tilde{\mathbf{L}}$ and construct a new $\mathbf{L}$ which does obey the right commutation relations. Because

$$\begin{aligned}\tilde{L}_i\hat{r}_j(f) &= \varepsilon_{ikl}x_k p_l\,(\hat{r}_j f) - q_e e\varepsilon_{ikl}x_k\hat{r}_j A_l f\\ &= -\mathrm{i}\varepsilon_{ikl}x_k\frac{\delta_{jl}}{r}f - \mathrm{i}\varepsilon_{ikl}x_k\hat{r}_j\partial_l f - q_e e\;\varepsilon_{ikl}x_k\hat{r}_j f A_l\\ &= \mathrm{i}\varepsilon_{ijk}\hat{r}_k f + \hat{r}_j\tilde{L}_i(f),\end{aligned} \tag{2.18}$$

which clearly implies $[\tilde{L}_i, \hat{r}_j] = \mathrm{i}\varepsilon_{ijk}\hat{r}_k$, we are naturally led to define the generalised orbital angular momentum as

$$\mathbf{L} = \tilde{\mathbf{L}} - \frac{q_e e q_m g}{4\pi}\hat{\mathbf{r}} = \tilde{\mathbf{L}} - \kappa\hat{\mathbf{r}} \tag{2.19}$$

since, in this case, the commutation relations are the desired

$$[L_i, L_j] = \mathrm{i}\varepsilon_{ijk}L_k. \tag{2.20}$$

Note the quantisation condition (2.12) implies the extra term $-\kappa\hat{\mathbf{r}}$ must take values $0, \pm 1/2, \pm 1, \ldots$. This is a bit odd: if a spinless particle with charge $+1$ is in the field of a Dirac monopole with minimal charge, such that $|\kappa| = 1/2$, the particle actually has generalised angular momentum $\ell = 1/2$! Conversely, a fermion has integer angular momentum. We will now see how the additional term $-\kappa\hat{r}$ is culpable for some thorny subtleties which arise when studying the scattering of fermions off monopoles.

To wrap up this section, let us revisit the chiral anomaly which arises when considering *massless* fermions. The axial current in this case is not conserved, precisely by

$$\partial_\mu J^\mu_A \sim \frac{\theta}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{2.21}$$

The approaching electron generates an electric field which, when anti-aligned with the magnetic field of the monopole, gives $\mathbf{E}\cdot\mathbf{B} \neq 0$ such that the theta-term (2.21) is non-vanishing. How is this related to the monopole? Well, it turns out the winding number n associated to the gauge transformations $\Omega \in U(1)$ (which leads to charge quantisation) is precisely the axial charge which is not conserved:

$$Q[\Omega] \sim \int d^4x\; F_{\mu\nu}\tilde{F}^{\mu\nu} \sim n. \tag{2.22}$$

The fact that, in the presence of the monopole, chirality is manifestly broken will prove important when determining what states are allowed to fall in or emerge out.

2.2 Solving the Dirac equation

Consider the quantum mechanics of a fermion with electric charge q_e and wavefunction $\Psi(x) = (\psi_L, \psi_R)^t$ in the presence of the Dirac monopole. The Hamiltonian of this system is

$$H = \gamma^0\boldsymbol{\gamma}\cdot(\mathbf{p} - q_e e\mathbf{A}) + \gamma^0 M \tag{2.23}$$

with $\mathbf{A}$ given by (2.4). The gamma matrices in the chiral representation are

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0\end{pmatrix}, \qquad \gamma^5 = -\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix}\mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2\end{pmatrix} \tag{2.24}$$

where $\sigma^\mu = (\mathbb{1}_2, \boldsymbol{\sigma})^\mu$ and $\bar{\sigma}^\mu = (\mathbb{1}_2, -\boldsymbol{\sigma})^\mu$ are Pauli matrices as usual. Explicitly, this gives

$$H = \begin{pmatrix} (\mathrm{i}\boldsymbol{\sigma}\cdot\nabla + q_e e\boldsymbol{\sigma}\cdot\mathbf{A}) & M\mathbb{1}_2 \\ M\mathbb{1}_2 & -(\mathrm{i}\boldsymbol{\sigma}\cdot\nabla + q_e e\boldsymbol{\sigma}\cdot\mathbf{A}) \end{pmatrix}. \tag{2.25}$$

We are interested in finding stationary states Ψ which obey

$$\mathrm{i}\partial_t\Psi = H\Psi = E\Psi. \tag{2.26}$$

Still taking inspiration from the solution of the hydrogen problem, let's look at the angular momentum. For an electrically charged Weyl fermion, we now have

$$\mathbf{J} = \mathbf{L} + \mathbf{S} = \tilde{\mathbf{L}} - \kappa\hat{\mathbf{r}} + \mathbf{S} \tag{2.27}$$

where $\mathbf{S} = \boldsymbol{\sigma}/2$ is the spin operator. For a Dirac fermion, $\mathbf{J}$ simply gets doubled on the diagonal so that

$$\mathbf{J}_{\text{Dirac}} = \begin{pmatrix} \mathbf{L} & 0 \\ 0 & \mathbf{L} \end{pmatrix} + \begin{pmatrix} \mathbf{S} & 0 \\ 0 & \mathbf{S} \end{pmatrix}. \tag{2.28}$$

In context it will be unambiguous whether we are dealing with a Dirac or Weyl fermion, so we will drop the label and either write $\mathbf{J} = \mathbf{L} + \frac{1}{2}\boldsymbol{\sigma}$ (for Weyl fermions) or $\mathbf{J} = \mathbf{L} + \frac{1}{2}\boldsymbol{\Sigma}$ (for Dirac fermions) where $\frac{1}{2}\boldsymbol{\Sigma} = \frac{1}{2}\,\mathrm{diag}(\boldsymbol{\sigma}, \boldsymbol{\sigma})$. To show $\mathbf{J}^2$ and, say, J_z are conserved, we must calculate the commutator of H with $\mathbf{J}$. Using the results from the previous section,

$$[H, \tilde{\mathbf{L}}] = -\mathrm{i}\gamma^0\boldsymbol{\gamma}\times(\mathbf{p} - q_e e\mathbf{A}) + \mathrm{i}\frac{\kappa}{r}\gamma^0\boldsymbol{\gamma} - \mathrm{i}\frac{\kappa}{r}\gamma^0(\boldsymbol{\gamma}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}, \tag{2.29}$$

$$[H, \hat{\mathbf{r}}] = -\mathrm{i}\frac{\kappa}{r}\gamma^0\boldsymbol{\gamma} + \mathrm{i}\frac{\kappa}{r}\gamma^0(\boldsymbol{\gamma}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}, \tag{2.30}$$

$$[H, \boldsymbol{\Sigma}] = +2\mathrm{i}\gamma^0\boldsymbol{\gamma}\times(\mathbf{p} - q_e e\mathbf{A}) \tag{2.31}$$

so that altogether the Hamiltonian commutes with $\mathbf{J}$. Defining $\boldsymbol{\pi} = \mathbf{p} - q_e e\mathbf{A}$, one can alternatively show this for the left- and right-handed components separately using $[\pi_i, L_j] = \mathrm{i}\varepsilon_{ijk}\pi_k$ which implies

$$\begin{aligned} [\boldsymbol{\sigma}\cdot\boldsymbol{\pi}, J_j] &= \sigma_i[\pi_i, L_j] + \frac{1}{2}[\sigma_i, \sigma_j]\pi_i \\ &= \mathrm{i}\varepsilon_{ijk}\sigma_i\pi_k + \mathrm{i}\varepsilon_{ijk}\sigma_k\pi_i = 0. \end{aligned} \tag{2.32}$$

This is the same result. In particular, since H commutes with $\mathbf{J}^2$ and J_z, we would like to express solutions to the wave equation (2.26) in terms of a simultaneous eigenbasis of all three operators. We will start by constructing the angular momentum eigenstates, on top of which appropriate radial functions will complete the derivation of the stationary states we're after.

Angular momentum eigenstates

Based on the analogy with the hydrogen problem, we can already suspect these functions will be modifications of spherical harmonics. If we first consider a particle without spin, the eigenfunctions of $\mathbf{J}^2 = \mathbf{L}^2$ and $J_z = L_z$ are the *generalised* or *monopole* spherical harmonics

$$Y_{\ell m\kappa}(\phi, \theta) = N_{\ell m\kappa}e^{\mathrm{i}(m+\kappa)\phi}\sqrt{1+x}^{m-\kappa}\sqrt{1-x}^{m+\kappa}\partial_x^{(\ell+m)}\left[(1+x)^{\ell+\kappa}(1-x)^{\ell-\kappa}\right]\Big|_{x=\cos\theta}. \tag{2.33}$$

The normalisation factor $N_{\ell m\kappa}$, along with a derivation of $Y_{\ell m\kappa}$, can be found in Appendix A. As with the potential $\mathbf{A}$ and generalised orbital momentum $\mathbf{L}$, the expression for $Y_{\ell m\kappa}$ depends on which region we consider. However, we can always relate functions in different regions by (2.13). Setting the magnetic charge to zero ($\kappa = 0$) we recover the usual spherical harmonics. The angular momentum projection along the z-axis, m, has the usual range

$$m = -\ell, -\ell+1, \ldots, \ell-1, \ell \tag{2.34}$$

whereas a novelty appears in the values of the integer ℓ which, instead of $\ell = 0, 1, 2, \ldots$, now spans

$$\ell = |\kappa|, |\kappa|+1, |\kappa|+2, \ldots\ . \tag{2.35}$$

When spin is added, such that $\mathbf{J} = \mathbf{L} + \mathbf{S}$, we must roll up our sleeves and deal with the details of addition of angular momentum. Since $\ell \geq s$ for a fermion with $s = 1/2$ in the vicinity of a non-trivial monopole, we have $|\ell - s| = \ell - s$ so that the total angular momentum can take values $j = \ell \pm s = \ell \pm 1/2$. We will denote the spin eigenstates of $\mathbf{S}^2$ and S_z by $|\frac{1}{2}\ m_s\rangle$, such that

$$|\frac{1}{2}\ \frac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\frac{1}{2}\ -\frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.36}$$

which have respective eigenvalues $s(s+1) = 3/4$ and $m_s = \pm 1/2$, and we will denote the generalised orbital momentum eigenstates $Y_{\ell m \kappa}$ by $|\ell\ m\rangle$. Then, the total eigenstates $|j\ m_j\rangle$ of $\mathbf{J}^2$ and J_z are

$$|j\ m_j\rangle = \sum_{\ell, m, m_s} |\ell\ m\rangle \otimes |\frac{1}{2}\ m_s\rangle\, \langle \ell\ m; \frac{1}{2}\ m_s | j\ m_j\rangle \tag{2.37}$$

in terms of Clebsch-Gordan coefficients $\langle \ell\ m; \frac{1}{2}\ m_s | j\ m_j\rangle$. Explicitly, we have constructed

$$|j\ m_j\rangle^{(1)} \equiv \mathcal{Y}^{(1)}_{jm_j\kappa}(\phi,\theta) = \begin{pmatrix} \sqrt{\frac{j+m_j}{2j}}\, Y_{(j-\frac{1}{2})(m_j-\frac{1}{2})\kappa}(\phi,\theta) \\ \sqrt{\frac{j-m_j}{2j}}\, Y_{(j-\frac{1}{2})(m_j+\frac{1}{2})\kappa}(\phi,\theta) \end{pmatrix} \qquad \text{if } j = \ell + \frac{1}{2}, \tag{2.38}$$

$$|j\ m_j\rangle^{(2)} \equiv \mathcal{Y}^{(2)}_{jm_j\kappa}(\phi,\theta) = \begin{pmatrix} -\sqrt{\frac{(j+1)-m_j}{2(j+1)}}\, Y_{(j+\frac{1}{2})(m_j-\frac{1}{2})\kappa}(\phi,\theta) \\ \sqrt{\frac{(j+1)+m_j}{2(j+1)}}\, Y_{(j+\frac{1}{2})(m_j+\frac{1}{2})\kappa}(\phi,\theta) \end{pmatrix} \qquad \text{if } j = \ell - \frac{1}{2} \tag{2.39}$$

which are called the generalised *spinor* harmonics and have the desired eigenvalues

$$\mathbf{J}^2 \mathcal{Y}_{jm_j\kappa}(\phi,\theta) = j(j+1)\mathcal{Y}_{jm_j\kappa}(\phi,\theta), \qquad J_z \mathcal{Y}_{jm_j\kappa}(\phi,\theta) = m_j \mathcal{Y}_{jm_j\kappa}. \tag{2.40}$$

A very important detail is that one can only define the functions

$$\begin{aligned} &\mathcal{Y}^{(1)}_{jm_j\kappa} \quad \text{when} \quad j = \ell + \frac{1}{2} = |\kappa| + \frac{1}{2}, |\kappa| + \frac{3}{2}, \ldots, \\ &\mathcal{Y}^{(2)}_{jm_j\kappa} \quad \text{when} \quad j = \ell - \frac{1}{2} = |\kappa| - \frac{1}{2}, |\kappa| + \frac{1}{2}, \ldots, \end{aligned}$$

which highlights that the dynamics of states with the lowest angular momentum value $j_0 \equiv |\kappa| - 1/2$ are special and can only be captured by $\mathcal{Y}^{(2)}_{jm_j\kappa}$. For example, a particle with unit electric charge approaching a minimal monopole ($\kappa = 1/2$) carrying lowest angular momentum $j_0 = 0$ must have a wavefunction with angular component

$$\mathcal{Y}^{(2)}_{00\frac{1}{2}}(\phi,\theta) = \begin{pmatrix} -\frac{1}{\sqrt{2}} Y_{\frac{1}{2}(-\frac{1}{2})\frac{1}{2}}(\phi,\theta) \\ +\frac{1}{\sqrt{2}} Y_{\frac{1}{2}(+\frac{1}{2})\frac{1}{2}}(\phi,\theta) \end{pmatrix} = -\frac{1}{2\sqrt{\pi}} \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}, \tag{2.41}$$

which is normalised as expected. States with $j = j_0$ will be distinguished and called *s-wave* states.

Turning our attention back to solving the Dirac equation (2.26), we would now like to understand how to express stationary states $\Psi(r,\phi,\theta)$ in terms of these angular eigenfunctions. The Hamiltonian involves the operator $(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})$ which is a generalisation of the standard helicity $(\boldsymbol{\sigma}\cdot\mathbf{p})$. Since we want a separation of variables of the form

$$\Psi(r,\phi,\theta) \sim \begin{pmatrix} f(r)\mathcal{Y}_{jm_j\kappa}(\phi,\theta) \\ g(r)\mathcal{Y}_{jm_j\kappa}(\phi,\theta)\end{pmatrix}, \tag{2.42}$$

we would like to know how this operator acts on the functions $\mathcal{Y}_{jm_j\kappa}$ we just derived. As previously noted, the helicity operator $(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})$ commutes with angular momentum such that its action on $\mathcal{Y}_{jm_j\kappa}$ must take the functions to the same eigenspace, whether that be $j = j_0$ or $j > j_0$.

Let us first consider states with $j > j_0$ which are described by both $\mathcal{Y}^{(1)}_{jm_j\kappa}$ and $\mathcal{Y}^{(2)}_{jm_j\kappa}$. We must have

$$\begin{aligned}(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})\mathcal{Y}^{(1)}_{jm_j\kappa} &= A_{11}\mathcal{Y}^{(1)}_{jm_j\kappa} + A_{12}\mathcal{Y}^{(2)}_{jm_j\kappa},\\ (\boldsymbol{\sigma}\cdot\boldsymbol{\pi})\mathcal{Y}^{(2)}_{jm_j\kappa} &= A_{21}\mathcal{Y}^{(1)}_{jm_j\kappa} + A_{22}\mathcal{Y}^{(2)}_{jm_j\kappa}.\end{aligned} \tag{2.43}$$

Finding the energy eigenstates boils down to determining the above matrix (A_{ij}) and then using the result to diagonalise H. To this end, we note that $\sigma_i\sigma_j = \delta_{ij} + \mathrm{i}\varepsilon_{ijk}\sigma_k$ gives the simple identity

$$(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}) = (\hat{\mathbf{r}}\cdot\boldsymbol{\pi}) + \mathrm{i}\boldsymbol{\sigma}\cdot(\hat{\mathbf{r}}\times\boldsymbol{\pi}) = (\hat{\mathbf{r}}\cdot\boldsymbol{\pi}) + \frac{\mathrm{i}}{r}(\boldsymbol{\sigma}\cdot\mathbf{L}) + \frac{\mathrm{i}}{r}\kappa(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}) \tag{2.44}$$

which leads to the useful trick

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}) = (\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})^2(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}) = (\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})(\hat{\mathbf{r}}\cdot\boldsymbol{\pi}) + \frac{\mathrm{i}}{r}(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})(\boldsymbol{\sigma}\cdot\mathbf{L}) + \frac{\mathrm{i}}{r}\kappa\mathbb{1}_2. \tag{2.45}$$

Since $\mathbf{A}\sim\hat{\boldsymbol{\phi}}$, when acting on spinor harmonics the first term vanishes, i.e.

$$(\hat{\mathbf{r}}\cdot\boldsymbol{\pi})\mathcal{Y}_{jm_j\kappa}(\phi,\theta) = (-\mathrm{i}\partial_r - q_e e A_r)\mathcal{Y}_{jm_j\kappa}(\phi,\theta) = 0, \tag{2.46}$$

while the second term in $(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})$ can be simplified using the simple relations

$$(\boldsymbol{\sigma}\cdot\mathbf{L})\mathcal{Y}^{(1)}_{jm_j\kappa} = \left(\mathbf{J}^2 - \mathbf{L}^2 - \frac{3}{4}\mathbb{1}_2\right)\mathcal{Y}^{(1)}_{jm_j\kappa} = \left(+j - \frac{1}{2}\right)\mathcal{Y}^{(1)}_{jm_j\kappa}, \quad \text{since } j = \ell + \frac{1}{2}, \tag{2.47}$$

$$(\boldsymbol{\sigma}\cdot\mathbf{L})\mathcal{Y}^{(2)}_{jm_j\kappa} = \left(\mathbf{J}^2 - \mathbf{L}^2 - \frac{3}{4}\mathbb{1}_2\right)\mathcal{Y}^{(2)}_{jm_j\kappa} = \left(-j - \frac{3}{2}\right)\mathcal{Y}^{(2)}_{jm_j\kappa}, \quad \text{since } j = \ell - \frac{1}{2}. \tag{2.48}$$

Putting everything together, we find

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})\mathcal{Y}^{(1)}_{jm_j\kappa} = \frac{\mathrm{i}}{r}\left[\left(+j-\frac{1}{2}\right)(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}) + \kappa\mathbb{1}_2\right]\mathcal{Y}^{(1)}_{jm_j\kappa}, \tag{2.49}$$

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})\mathcal{Y}^{(2)}_{jm_j\kappa} = \frac{\mathrm{i}}{r}\left[\left(-j-\frac{3}{2}\right)(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}) + \kappa\mathbb{1}_2\right]\mathcal{Y}^{(2)}_{jm_j\kappa}. \tag{2.50}$$

We now just need to find out how the spin projection $(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})$ acts on the spinor harmonics. One finds

$$[\mathbf{L}^2, \boldsymbol{\sigma}\cdot\hat{\mathbf{r}}] = +2\boldsymbol{\sigma}\cdot\hat{\mathbf{r}} + 2\mathrm{i}\,(\boldsymbol{\sigma}\times\hat{\mathbf{r}})\cdot\mathbf{L}, \qquad [L_z, \boldsymbol{\sigma}\cdot\hat{\mathbf{r}}] = +\mathrm{i}\,(\boldsymbol{\sigma}\times\hat{\mathbf{r}})_z\,, \tag{2.51}$$

$$[\mathbf{L}\cdot\sigma, \boldsymbol{\sigma}\cdot\hat{\mathbf{r}}] = -2\boldsymbol{\sigma}\cdot\hat{\mathbf{r}} - 2\mathrm{i}\,(\boldsymbol{\sigma}\times\hat{\mathbf{r}})\cdot\mathbf{L}, \qquad [\sigma_z, \boldsymbol{\sigma}\cdot\hat{\mathbf{r}}] = -2\mathrm{i}\,(\boldsymbol{\sigma}\times\hat{\mathbf{r}})_z \tag{2.52}$$

such that the operator $(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})$ also commutes with $\mathbf{J}^2$ and J_z. By the exact same argument as for $(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})$, the action of $(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})$ must then take spinor harmonics to linear combinations

$$\begin{aligned}(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\,\mathcal{Y}^{(1)}_{jm_j\kappa} &= B_{11}\mathcal{Y}^{(1)}_{jm_j\kappa} + B_{12}\mathcal{Y}^{(2)}_{jm_j\kappa},\\ (\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\,\mathcal{Y}^{(2)}_{jm_j\kappa} &= B_{21}\mathcal{Y}^{(1)}_{jm_j\kappa} + B_{22}\mathcal{Y}^{(2)}_{jm_j\kappa}.\end{aligned} \tag{2.53}$$

In particular, because $(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})$ commutes with J_z, the coefficients B_{ij} must be independent of m_j. In Appendix A, the calculation is performed, leveraging this freedom by specifying $m_j=-j$, and gives

$$B_{11} = -B_{22} = -\frac{\kappa}{(j+\frac{1}{2})}, \qquad B_{12} = B_{21} = -\frac{\mu}{(j+\frac{1}{2})}. \tag{2.54}$$

Substituting these results into (2.47), we find

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})\mathcal{Y}^{(1)}_{jm_j\kappa} = \frac{\mathrm{i}}{r}\left[\frac{\kappa}{(j+\frac{1}{2})}\mathcal{Y}^{(1)}_{jm_j\kappa} - \frac{\mu(j-\frac{1}{2})}{(j+\frac{1}{2})}\mathcal{Y}^{(2)}_{jm_j\kappa}\right] \tag{2.55}$$

with a similar equation for $\mathcal{Y}^{(2)}_{jm_j\kappa}$ so that, in total,

$$A_{11} = -A_{22} = \frac{\mathrm{i}}{r}\frac{\kappa}{(j+\frac{1}{2})}, \qquad A_{12} = -\frac{\mathrm{i}}{r}\frac{\mu(j-\frac{1}{2})}{(j+\frac{1}{2})}, \qquad A_{21} = -\frac{\mathrm{i}}{r}\frac{\mu(-j-\frac{3}{2})}{(j+\frac{1}{2})}. \tag{2.56}$$

Clearly the operator $(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})$ mixes the spinor harmonic functions $\mathcal{Y}_{jm_j\kappa}$, which means it will be useful to find a rotated basis of angular eigenfunctions

$$\xi^{(1)}_{jm_j\kappa} = \cos\alpha\mathcal{Y}^{(1)}_{jm_j\kappa} - \sin\alpha\mathcal{Y}^{(2)}_{jm_j\kappa}, \tag{2.57}$$

$$\xi^{(2)}_{jm_j\kappa} = \sin\alpha\mathcal{Y}^{(1)}_{jm_j\kappa} + \cos\alpha\mathcal{Y}^{(2)}_{jm_j\kappa} \tag{2.58}$$

which we will ask to be more compatible with $(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})$, and thus with H. Since we are looking for solutions $\Psi_{j,E}$ of the form

$$H\Psi_{j,E}(r,\phi,\theta) \sim H\begin{pmatrix} f(r)\xi^{(1)}_{jm_j\kappa}(\phi,\theta)\\ g(r)\xi^{(2)}_{jm_j\kappa}(\phi,\theta)\end{pmatrix} = E\begin{pmatrix} f(r)\xi^{(1)}_{jm_j\kappa}(\phi,\theta)\\ g(r)\xi^{(2)}_{jm_j\kappa}(\phi,\theta)\end{pmatrix}, \tag{2.59}$$

it will also be momentarily convenient to work in the Dirac basis $\gamma^\mu_{\text{Dirac}} = U\gamma^\mu_{\text{chiral}}U^{-1}$ where

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} \implies H_{\text{Dirac}} = \begin{pmatrix} M\mathbb{1}_2 & \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\\ \boldsymbol{\sigma}\cdot\boldsymbol{\pi} & -M\mathbb{1}_2\end{pmatrix} \tag{2.60}$$

so that, for some unspecified radial functions, we can simply require the natural action

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})f(r)\xi^{(1)} \sim \tilde{f}(r)\xi^{(2)}, \qquad (\boldsymbol{\sigma}\cdot\boldsymbol{\pi})g(r)\xi^{(2)} \sim \tilde{g}(r)\xi^{(1)}. \tag{2.61}$$

Had we been working in the chiral basis, it's clear the above action would have been more complicated due to which entry of the Hamiltonian carries $(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})$. Our job is now to find the angle α responsible for the change of basis (2.57) & (2.58). Looking at the trick (2.45), since

$$(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}) = \begin{pmatrix}\cos\theta & \sin\theta e^{-\mathrm{i}\phi}\\ \sin\theta e^{+\mathrm{i}\phi} & -\cos\theta\end{pmatrix}, \tag{2.62}$$

we see the first term only acts as $(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})(-\mathrm{i}\partial_r) = (-\mathrm{i}\partial_r)(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})$ on, say, $f(r)\xi^{(1)}$, so we then need $(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})$ to flip $\xi^{(1)}$ to $\pm\xi^{(2)}$ and vice versa. (No other factor complies with $(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})^2 = \mathbb{1}_2$.) This means α satisfies

$$(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\left(\cos\alpha\mathcal{Y}^{(1)}_{jm_j\kappa} - \sin\alpha\mathcal{Y}^{(2)}_{jm_j\kappa}\right) = \pm\left(\sin\alpha\mathcal{Y}^{(1)}_{jm_j\kappa} + \cos\alpha\mathcal{Y}^{(2)}_{jm_j\kappa}\right) \tag{2.63}$$

Using the explicit action (2.54) of $(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})$ on $\mathcal{Y}_{jm_j\kappa}$, a simple calculation yields

$$\tan\alpha = \frac{\mp(j+\frac{1}{2})-\mu}{\kappa} = \frac{\kappa}{\mp(j+\frac{1}{2})+\mu}. \tag{2.64}$$

It ends up being simpler to choose the lower sign. A more symmetric expression is

$$\tan\alpha = \frac{\sqrt{(j+\frac{1}{2})+\kappa} - \sqrt{(j+\frac{1}{2})-\kappa}}{\sqrt{(j+\frac{1}{2})+\kappa} + \sqrt{(j+\frac{1}{2})-\kappa}}. \tag{2.65}$$

which, after normalisation, leads to the result we were after: the basis rotation

$$\sin\alpha = \frac{1}{2}\frac{\kappa}{|\kappa|}\frac{\sqrt{(j+\frac{1}{2})+\kappa} - \sqrt{(j+\frac{1}{2})-\kappa}}{\sqrt{j+\frac{1}{2}}}, \qquad \cos\alpha = \frac{1}{2}\frac{\kappa}{|\kappa|}\frac{\sqrt{(j+\frac{1}{2})+\kappa} + \sqrt{(j+\frac{1}{2})-\kappa}}{\sqrt{j+\frac{1}{2}}}. \tag{2.66}$$

The factor of $\kappa/|\kappa|$ is inserted so the overall wavefunction normalisation is convenient. Using the expressions (2.56) for A_{ij} and (2.64) for $\tan\alpha$, we can now reap the fruit of our labour by calculating

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})\xi^{(1)}_{jm_j\kappa} = \frac{\mathrm{i}}{r}\,(1-\mu)\,\xi^{(2)}, \qquad (\boldsymbol{\sigma}\cdot\boldsymbol{\pi})\xi^{(2)}_{jm_j\kappa} = \frac{\mathrm{i}}{r}\,(1+\mu)\,\xi^{(1)}. \tag{2.67}$$

Finally, we are ready to act with the Hamiltonian and solve for eigenstates. We find, for example,

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})f(r)\xi^{(1)} = (-\mathrm{i}\partial_r)f(r)(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\xi^{(1)} + f(r)(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})\xi^{(1)} = \mathrm{i}\left(\partial_r + \frac{1}{r}(1-\mu)\right)f(r)\xi^{(2)} \tag{2.68}$$

and similarly

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})g(r)\xi^{(2)} = \mathrm{i}\left(\partial_r + \frac{1}{r}(1+\mu)\right)g(r)\xi^{(1)}. \tag{2.69}$$

This means the radial functions in (2.59) satisfy

$$(M-E)f(r) = -\mathrm{i}\left(\partial_r + \frac{1}{r}(1+\mu)\right)g(r), \tag{2.70}$$

$$(M+E)g(r) = +\mathrm{i}\left(\partial_r + \frac{1}{r}(1-\mu)\right)f(r) \tag{2.71}$$

which imply, making the judicious definition $f(r) = F(r)/\sqrt{r}$, $g(r) = G(r)/\sqrt{r}$,

$$\left[r^2\partial_r^2 + r\partial_r + r^2(E^2-M^2) - \left(\mu+\frac{1}{2}\right)^2\right]G(r) = 0, \tag{2.72}$$

$$\left[r^2\partial_r^2 + r\partial_r + r^2(E^2-M^2) - \left(\mu-\frac{1}{2}\right)^2\right]F(r) = 0. \tag{2.73}$$

We have stumbled on Bessel equations, and these are solved by $F(r) = c_1 J_{\mu-\frac{1}{2}}(kr)$, $G(r) = c_2 J_{\mu+\frac{1}{2}}(kr)$ for some constants c_1, c_2, where we recognise $k^2 = E^2 - M^2$ as the fermion's momentum. The relative normalisation between the spin-up and spin-down components is fixed by the identity

$$\partial_x \left(x^n J_n(x)\right) = x^n J_{n-1}(x) \implies \left(\frac{1}{k}\partial_r + \frac{\mu+\frac{1}{2}}{r}\right) J_{\mu+\frac{1}{2}}(kr) = J_{\mu-\frac{1}{2}}(kr) \tag{2.74}$$

which, in tandem with (2.70), forces

$$c_1 = \frac{\mathrm{i}k}{E-M} c_2. \tag{2.75}$$

Thus, up to overall normalisation, solutions to (2.59) have radial components

$$f(r) = \frac{\mathrm{i}k}{E-M}\frac{1}{\sqrt{kr}} J_{\mu-\frac{1}{2}}(kr), \quad g(r) = \frac{1}{\sqrt{kr}} J_{\mu+\frac{1}{2}}(kr). \tag{2.76}$$

However, the labels of $\xi^{(1)}$ and $\xi^{(2)}$ are simply conventional. We could equally well have solved for

$$\Psi_{j,E}(r,\phi,\theta) \sim \begin{pmatrix} g(r)\xi^{(2)}(\phi,\theta) \\ f(r)\xi^{(1)}(\phi,\theta) \end{pmatrix} \tag{2.77}$$

for which instead, noting the sign flip in the mass prefactor, the radial functions are

$$f(r) = \frac{\mathrm{i}k}{E+M}\frac{1}{\sqrt{kr}} J_{\mu-\frac{1}{2}}(kr), \quad g(r) = \frac{1}{\sqrt{kr}} J_{\mu+\frac{1}{2}}(kr). \tag{2.78}$$

In summary, we just found that Dirac fermions with angular momentum $j > j_0$ can be expressed in terms of the general energy eigenstates (in the Dirac basis) as

$$\Psi_{j,E}(r,\phi,\theta) = \sum_{m_j=-j}^{j} \frac{1}{\sqrt{kr}} \begin{pmatrix} \frac{\mathrm{i}k}{E-M} a J_{\mu-\frac{1}{2}}(kr)\xi^{(1)}_{jm_j\kappa}(\phi,\theta) + b J_{\mu+\frac{1}{2}}(kr)\xi^{(2)}_{jm_j\kappa}(\phi,\theta) \\ \frac{\mathrm{i}k}{E+M} b J_{\mu-\frac{1}{2}}(kr)\xi^{(1)}_{jm_j\kappa}(\phi,\theta) + a J_{\mu+\frac{1}{2}}(kr)\xi^{(2)}_{jm_j\kappa}(\phi,\theta) \end{pmatrix}. \tag{2.79}$$

What about s-wave states which have $j = j_0$? Unlike states with $j > j_0$, for which we needed to determine the linear combinations (2.43), this time our job is much easier since we just need to know how $(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})$ acts on the single available angular eigenfunction

$$\eta_{m_j}(\phi,\theta) \equiv \mathcal{Y}^{(2)}_{j_0 m_j \kappa}(\phi,\theta). \tag{2.80}$$

Reusing the expression (2.50), the helicity acts as

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})\eta_{m_j} = \frac{\mathrm{i}}{r}\left[\left(-j_0 - \frac{3}{2}\right)(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}) + \kappa \mathbb{1}_2\right]\eta_{m_j} \tag{2.81}$$

and we once again set $m_j = -j_0$ (see (A.22)) to find

$$(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\eta_{m_j} = \frac{\kappa}{|\kappa|}\eta_{m_j}. \tag{2.82}$$

After substituting into (2.45) this implies $(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}) f(r)\eta_{m_j} = -\mathrm{i}\frac{\kappa}{|\kappa|}\left(\partial_r + \frac{1}{r}\right) f(r)\eta_{m_j}$, so that the radial components in energy eigenstates of the form $\Psi_{j_0,E}(r,\phi,\theta) \sim \left(f(r)\eta_{m_j}(\phi,\theta),\ g(r)\eta_{m_j}(\phi,\theta)\right)$ in the Dirac basis must satisfy

$$(M-E)f(r) - \mathrm{i}\frac{\kappa}{|\kappa|}\left(\partial_r + \frac{1}{r}\right) g(r) = 0, \qquad (M+E)g(r) + \mathrm{i}\frac{\kappa}{|\kappa|}\left(\partial_r + \frac{1}{r}\right) f(r) = 0. \tag{2.83}$$

Equivalently, this time making the substitutions $f(r) = F(r)/r$ and $g(r) = G(r)/r$,

$$(M - E)F(r) = \mathrm{i}\frac{\kappa}{|\kappa|}\partial_r G(r), \qquad (M + E)G(r) = -\mathrm{i}\frac{\kappa}{|\kappa|}\partial_r F(r). \tag{2.84}$$

If we impose the boundary condition $F(0) = 0$, these differential equations are easily solved by

$$F(r) = \frac{\kappa}{|\kappa|}\sin(kr), \qquad G(r) = -\frac{\mathrm{i}k}{E + M}\cos(kr). \tag{2.85}$$

up to some normalisation constant.Thus, in the chiral basis an s-wave with energy E has wavefunction

$$\Psi_{j_0,E}(r,\phi,\theta) = \frac{\mathrm{i}}{\sqrt{2}}\sum_{m_j=-j_0}^{j_0}\frac{1}{r}\begin{pmatrix} \left(\frac{k}{E+M}\cos(kr) - \mathrm{i}\frac{\kappa}{|\kappa|}\sin(kr)\right)\eta_{m_j}(\phi,\theta) \\ -\left(\frac{k}{E+M}\cos(kr) + \mathrm{i}\frac{\kappa}{|\kappa|}\sin(kr)\right)\eta_{m_j}(\phi,\theta) \end{pmatrix}. \tag{2.86}$$

In the end, we can say a Dirac fermion in the presence of a point monopole can be expressed in terms of the superposition of different energy and angular momentum modes as

$$\Psi(t,\mathbf{x}) = \int_0^\infty dE\left[\Psi_{j_0,E}(\mathbf{x}) + \sum_{j>j_0}\Psi_{j,E}(\mathbf{x})\right]e^{-\mathrm{i}Et}. \tag{2.87}$$

Having derived the spectrum of states, what can we say about incoming and outgoing scattering states? For simplicity, assume $\kappa > 0$ and take the fermion to be massless. This entails $k = E$ such that in this case the angular momentum modes are, up to normalisation,

$$\Psi_{j,E} \sim \sum_{m_j}\frac{1}{\sqrt{r}}\begin{pmatrix} \mathrm{i}J_{\mu-\frac{1}{2}}(Er)\xi^{(1)}_{jm_j\kappa} - J_{\mu+\frac{1}{2}}(Er)\xi^{(2)}_{jm_j\kappa} \\ \mathrm{i}J_{\mu-\frac{1}{2}}(Er)\xi^{(1)}_{jm_j\kappa} + J_{\mu+\frac{1}{2}}(Er)\xi^{(2)}_{jm_j\kappa} \end{pmatrix}, \tag{2.88}$$

$$\Psi_{j_0,E} \sim \sum_{m_j}\frac{1}{r}\begin{pmatrix} e^{-\mathrm{i}Er}\eta_{m_j} \\ -e^{+\mathrm{i}Er}\eta_{m_j} \end{pmatrix}. \tag{2.89}$$

At large r, the left-handed and right-handed components of the solutions (2.88) for $j > j_0$ both have incoming *and* outgoing modes. This is because the Bessel functions have the asymptotic expansion

$$J_{\mu+\frac{1}{2}}(x) \sim \frac{\sin(x)}{\sqrt{x}}, \qquad J_{\mu-\frac{1}{2}}(x) \sim \frac{\cos(x)}{\sqrt{x}} \qquad \text{for } x \to \infty. \tag{2.90}$$

In contrast, we see that for the lowest angular momentum modes (s-wave fermions) (2.89), the left-handed and right-handed modes are distinctly incoming *or* outgoing wavepackets! We can write

$$\Psi_{\text{s-wave}} = \int_0^\infty dE\ \Psi_{j_0,E}(\mathbf{x})e^{-\mathrm{i}Et} = \sum_{m_j}\frac{1}{r}\begin{pmatrix} \chi_{\text{in}}(t+r)\eta_{m_j} \\ \chi_{\text{out}}(t-r)\eta_{m_j} \end{pmatrix} \tag{2.91}$$

for Weyl spinors $\psi_L \sim \chi_{\text{in}}(t+r)\eta_{m_j}/r$ and $\psi_R \sim \chi_{\text{out}}(t-r)\eta_{m_j}/r$ [9]. If we assume $q_m > 0$, then for $\kappa > 0$ the ingoing left-handed (right-handed) modes have negative (positive) charge, while the outgoing left-handed (right-handed) modes have positive charge. This is summarised in Table 1.

Had we not been in such a hurry to find the s-wave energy eigenstates, we could have spotted this peculiar feature much earlier—when we derived (2.82). We see that the spin projection toward the monopole (on the angular part of the wavefunction) of an s-wave fermion is independent of r, and evaluates to the sign of the charge product κ. A striking consequence is the following. Suppose we wanted to study, say, a right-handed electron in the field of the monopole with positive magnetic charge, such that $\kappa < 0$. Because it is right-handed, if the electron was incoming then its spin projection would be $(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}) < 0$ (see Figure 2). This is perfectly compatible with (2.82). In contrast, a right-handed electron *cannot* be outgoing since that would correspond to $(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}) > 0$ which, before even analysing the spectrum of the system, is forbidden by (2.82)! The origin of this bizarre behaviour is the additional term $-\kappa\hat{\mathbf{r}}$ due to the monopole.

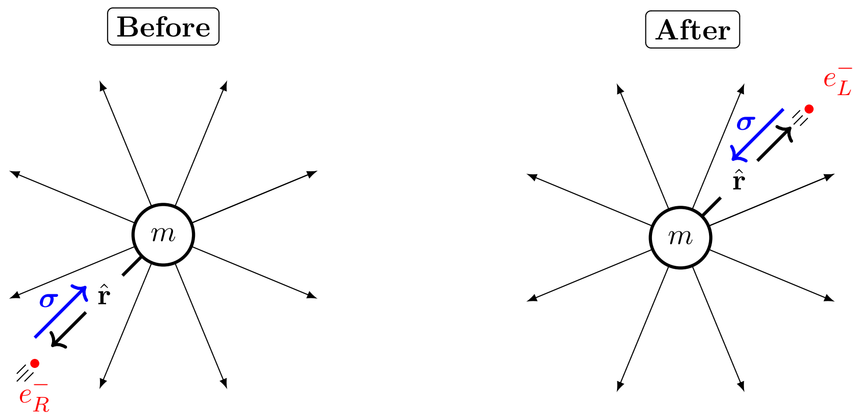

Figure 2. S-waves exhibit the peculiar property that $(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}})$ is constant and equal to $\mathrm{sgn}(\kappa)$, which implies the helicity of charged fermions with lowest angular momentum flips after passing the monopole.

However, an outgoing left-handed electron e_L^- or right-handed postrion e_R^+ would preserve $(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}) > 0$ and so are candidate outgoing states. Looking at Table 1, we see there are thus two options for the scattering of a right-handed electron off a monopole:

$$e_R^- + m \longrightarrow e_L^- + m, \qquad \text{or} \qquad e_R^- + m \longrightarrow e_R^+ + m. \tag{2.92}$$

This means for an s-wave massless fermion approaching and passing through a monopole, either

(1) the monopole flips the helicity of the fermion, or

(2) the monopole switches the sign of the fermion's electric charge.

The first option entails violation of chirality, which is otherwise a good symmetry of the Hamiltonian since $[H, \Sigma \cdot \boldsymbol{\pi}] = 0$. However, this is fake news due to the ABJ anomaly, and so we can't protest about helicity being flipped. The second option implies the monopole absorbs some electric charge. This proves more problematic as for the pointlike monopole we discussed, the charge residing on the monopole would lead to a divergent Coulomb energy.

Even in the massless limit, there is an asymmetry between the left- and right-handed fermions. How come? The answer lies in (2.21). While the massless Dirac fermion we are considering enjoys the axial symmetry *classically*, such that chirality is conserved, we can't ask the same of the quantum theory because of the ABJ chiral anomaly. The bizarre scattering process (2.92) is nothing but an example which explicitly points this out.

For either option (1) or (2), we need to impose a suitable boundary condition at the monopole core ($r = 0$). We will soon see this really comes from the low-energy description of a 't Hooft-Polyakov/non-Abelian/non-singular monopole, which has finite size in contrast to the Dirac/Abelian/singular point monopole we have studied so far.

Why is the s-wave special?

As opposed to the $j > j_0$ modes, the s-waves do not feature elastic scattering: there is a 100% chance of incoming states hitting the core and coming out transformed. Why is this the case?

In the quantum mechanical picture we took above, we can explain this by looking at the wavefunction at $r = 0$. Owing to asymptotic expansions of the Bessel functions, the wavefunction has the form $\Psi_{j,E} \sim r^{\mu-1}$ near the origin. The probability of finding this fermion at the monopole core thus goes like $r^2|\Psi_{j,E}|^2 \sim r^{2\mu}$ since we need to account for the spherical integration measure. Only for $j = j_0$ does this allow for the fermion to hit the core. This reflects that the radial equations of s-wave states don't feature a centrifugal barrier, unlike those for $j > j_0$

The Dirac Hamiltonian (2.25) and Schrödinger equation can also be rewritten as the usual Dirac equation for the *field* $\Psi_{j,E}$. In this field theory interpretation, the equivalent statement is that the flow of charge along the radial direction (into the monopole) is non-zero for s-wave states [10]:

$$\lim_{r\to 0} 4\pi r^2 \overline{\Psi}\hat{\mathbf{r}} \cdot \boldsymbol{\gamma}\Psi \neq 0. \tag{2.93}$$

As pointed out in [11], the statement that s-waves alone are special is a simplification. There are added subtleties when considering non-Abelian monopoles which we won't discuss.

2.3 Non-Abelian monopoles

So far we have been scattering fermions of the monopoles characterised by the Dirac vector potential (2.14) which lives in the Abelian gauge group $U(1)$. What about non-Abelian groups?

't Hooft-Polyakov monopoles

For example, let us work with the simplest example of $SU(2)$. We will take the matter content to be N Weyl fermions ψ_i transforming in the fundamental representation $\mathbf{2}$. Denote the generators of $SU(2)$ by $T^a = \tau^a/2$ where τ^a are the Pauli matrices acting on $SU(2)$ indices for $a = 1, 2, 3$. The model is

$$\mathcal{L} = -\frac{1}{4}G^{a,\mu\nu}G^a_{\mu\nu} - \frac{1}{2}D^\mu\varphi^a D_\mu\varphi^a - V(|\varphi|) \tag{2.94}$$

where the gauge fields are $W_\mu = W^a_\mu T^a$ with non-Abelian field strength tensor

$$G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + \mathrm{i}\tilde{g}\varepsilon^{abc}W^b_\mu W^c_\nu, \tag{2.95}$$

and the Higgs triplet $\varphi = \varphi^a T^a$ is subject to a potential $V(|\varphi|)$ and covariant derivative given by

$$V(|\varphi|) = \frac{1}{2}\tilde{\mu}^2|\varphi|^2 + \frac{1}{8}\lambda|\varphi|^4, \tag{2.96}$$

$$D_\mu\varphi^a = \partial_\mu\varphi^a - \mathrm{i}\tilde{g}W^b_\mu T^b(\mathrm{adj})\left[\varphi^a\right]. \tag{2.97}$$

At low enough energies, the Higgs acquires a vacuum expectation value $\langle\varphi\rangle^2 = -2\tilde{\mu}^2/\lambda \equiv v^2$ determined by $V'(\langle\varphi\rangle) = 0$. For convenience we can rotate the Higgs in isospin space to lie along the z-axis by choice of gauge, such that $\langle\varphi\rangle = 2vT^3 = v\tau^3$. In this gauge, which we will call the *unitary gauge*, the unbroken gauge group is generated by T^3 since it's the only linear combination of generators of $\mathfrak{su}(2)$ which leaves $\langle\varphi\rangle$ untouched under adjoint action. Substituting the fluctuation around the vacuum $\varphi = 2(v + h(x))T^3$ into the kinetic term, we find

$$\begin{aligned} D^\mu\varphi^a D_\mu\varphi^a &= \partial^\mu\varphi^a\partial_\mu\varphi^a + 2\tilde{g}\varepsilon^{abc}W^b_\mu\varphi^c\partial^\mu\varphi^a + \tilde{g}^2\varepsilon^{abc}\varepsilon^{ade}W^b_\mu\varphi^c W^{d,\mu}\varphi^e \\ &= \partial_\mu h(x)\partial^\mu h(x) + \tilde{g}^2(v + h(x))^2\left[W^1_\mu W^{1,\mu} + W^2_\mu W^{2,\mu}\right] \end{aligned} \tag{2.98}$$

which demonstrates the gauge fields $W_\mu^\pm = (W_\mu^1 \mp \mathrm{i} W_\mu^2)/\sqrt{2}$ acquire a mass $m_W = v\tilde{g}$ via Higgsing while the photon is $A_\mu = W_\mu^3$ and remains massless. This corresponds to the spontaneous symmetry breaking $SU(2) \longrightarrow U(1)_{\text{eff}}$ where the effective electromagnetism group $U(1)_{\text{eff}}$ is generated by $Q = \tau^3/2$. We thus see that the N fundamental Weyl fermions have been split into two groups; N ψ_i^1 with charge $+\tilde{g}/2$ and N ψ_i^2 with charge $-\tilde{g}/2$. We can identify these with electric charges if the winding of $U(1)_{\text{eff}}$ is taken to be $n = 1$, and $\tilde{g} = 4\pi/q_m g$ such that ψ_i^1 are charged with $+q_e$ and ψ_i^2 are charged with $-q_e$.

't Hooft [12] and Polyakov [13] found that, hidden in the classical Yang-Mills field equations for this $SU(2)$ theory, there is a solution describing a static, spherically symmetric magnetic monopole. In arbitrary gauge, it is described by

$$A_0(r) = 0, \quad A_i(r) = \frac{1}{2\mathrm{i}} \varepsilon^{iab} \hat{r}^a \tau^b \left[1 - A(r)\right], \qquad \varphi^a(r) = v\tau^a \hat{r}^a \left[1 - f(r)\right] \tag{2.99}$$

where $\mathbf{r}$ is the radial vector (and specifies the direction of φ in $SU(2)$ space) and the functions satisfy

$$A(0), f(0) \to 1, \qquad A(\infty), f(\infty) \to 0. \tag{2.100}$$

Thus, far away from the monopole, the fields asymptote to

$$\lim_{r\to\infty} A_i(r) = \frac{1}{2\mathrm{i}} \varepsilon^{iab} \hat{r}^a \tau^b, \qquad \varphi^a(r) = v\tau^a \hat{r}^a. \tag{2.101}$$

We can bring these fields to the unitary gauge via the transformation $\Omega(\mathbf{x}) \in SU(2)$ given by

$$\Omega(\phi, \theta) = e^{-\mathrm{i}\frac{\phi}{2}\tau^3} e^{+\mathrm{i}\frac{\theta}{2}\tau^2} e^{+\mathrm{i}\frac{\phi}{2}\tau^3} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} e^{-\mathrm{i}\phi} \\ -\sin\frac{\theta}{2} e^{+\mathrm{i}\phi} & \cos\frac{\theta}{2} \end{pmatrix}, \tag{2.102}$$

which acts as

$$\begin{gathered} \Omega\tau^1\Omega^{-1} = \begin{pmatrix} \sin\theta\cos\phi & \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} e^{-2\mathrm{i}\phi} \\ \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} e^{+2\mathrm{i}\phi} & -\sin\theta\cos\phi \end{pmatrix}, \qquad \Omega\tau^3\Omega^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta e^{-\mathrm{i}\phi} \\ -\sin\theta e^{\mathrm{i}\phi} & -\cos\theta \end{pmatrix}, \\ \Omega\tau^2\Omega^{-1} = \begin{pmatrix} \sin\theta\sin\phi & -\mathrm{i}\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} e^{-2\mathrm{i}\phi}\right) \\ \mathrm{i}\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} e^{+2\mathrm{i}\phi}\right) & -\sin\theta\sin\phi \end{pmatrix} \end{gathered} \tag{2.103}$$

so that, in total, this transformation results in $\hat{r}^a\tau^a \longrightarrow \Omega\hat{r}^a\tau^a\Omega^{-1} = \tau^3$. In this gauge the Higgs indeed points along the z-axis in $SU(2)$. Meanwhile, also noting derivatives like

$$\Omega^{-1}\partial_z\Omega = \frac{\sin\theta}{2r}\left(\cos\phi\tau^1 + \sin\phi\tau^2\right), \qquad \frac{1}{r\sin\theta}\Omega^{-1}\partial_\phi\Omega = \frac{1}{r\sin\theta}\begin{pmatrix} \cos^2\frac{\theta}{2} + \mathrm{i}\sin^2\frac{\theta}{2} & -\frac{1}{2}\sin\theta e^{-\mathrm{i}\phi}(1+\mathrm{i}) \\ \frac{1}{2}\sin\theta e^{+\mathrm{i}\phi}(1-\mathrm{i}) & \cos^2\frac{\theta}{2} - \mathrm{i}\sin^2\frac{\theta}{2} \end{pmatrix}$$

we find the 't Hooft-Polyakov vector potential far away from the monopole, taken to unitary gauge, is

$$\mathbf{A} \longrightarrow \mathbf{A}' = \Omega\mathbf{A}\Omega^{-1} - \frac{\mathrm{i}}{\tilde{g}}\Omega^{-1}\boldsymbol{\nabla}\Omega = \frac{1}{\tilde{g}}\frac{1-\cos\theta}{r\sin\theta}\hat{\boldsymbol{\phi}}. \tag{2.104}$$

If we make the identification $q_m g = 4\pi/\tilde{g}$, we seem to have magically retrieved the Dirac monopole! Is this magic? There is a physics explanation for this (although physics can be magical sometimes). At low enough energies, or equivalently at long enough range, the gauge group breaks down as $SU(2) \longrightarrow U(1)_{\text{eff}}$, whereby the Higgs triplet VeV leads to a potential which confines the residual $SU(2)$ degrees of freedom to a core of size m_W^{-1}. Thus, fermions far away from the monopole only feel $U(1)_{\text{eff}}$ corresponding to $\mathbf{A}'$ and it's only once they make contact with the core that the $SU(2)$ physics comes into play. The incoming fermion can excite the residual degrees of freedom trapped inside the core, which then radiate the added energy in the form of outgoing fermions. This dyonic excitation is precisely captured by the boundary conditions we will impose. In the modern language of effective field theory, we can view the $SU(2)$ gauge theory as the UV completion of $U(1)_{\text{eff}}$ [9, 11].

For $SU(2)$, there is once again an additional term to the angular momentum, which this time is promoted to the non-Abelian version $\mathbf{T}$. In the unitary gauge, and with the above identification we recover $-\kappa\hat{\mathbf{r}}$ [7, 10].

Grand unified theories

The above analysis of $SU(2)$ is extremely important for the following reason. There is an alternative, more realistic approach in which we find a monopole in the $U(1)_{\text{eff}}$ to which the Standard Model gauge group $G = SU(3)_c \times SU(2)_L \times U(1)_Y$ breaks. In particular, theorists suspect the Standard Model itself is the low-energy effective description of a grand unified theory (GUT) with gauge group $\mathcal{G}$ which spontaneously breaks to G. It is accurate to consider the Dirac monopole as static because of how heavy $SU(5)$ gauge bosons X are. An estimate is $m_x \sim 10^{15}$ GeV.

The $SU(5)$ gauge groups breaks to the everyday gauge fields in two steps, but results in the everday

$$SU(5) \longrightarrow SU(3)_c \times SU(2)_L \times U(1)_Y \longrightarrow SU(3)_c \times U(1)_{\text{em}}. \tag{2.105}$$

The lingering $SU(3)_c$ endows the monopole with a *colour* magnetic charge which is why, as we mentioned in Section 2.1, quarks satisfy a different quantisation condition. We can make the natural choice of embedding of $SU(3)_c \times SU(2)_L$into $SU(5)$ as

$$SU(3)_c \hookrightarrow \begin{pmatrix} SU(3)_c & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}, \qquad SU(2)_L \hookrightarrow \begin{pmatrix} \mathbb{1}_3 & 0 \\ 0 & SU(2)_L \end{pmatrix} \tag{2.106}$$

where we are agnostic about the off-diagonal actions of $SU(5)$. With this natural choice of decomposition, we can populate the matter content of the Standard Model with, taking for simplicity just the 1st generation, the fermions transforming in the representations

$$\bar{\mathbf{5}} = \begin{pmatrix} \overline{d}_1 \\ \overline{d}_2 \\ \overline{d}_3 \\ e^- \\ \nu_e \end{pmatrix}, \qquad \mathbf{10} = \begin{pmatrix} 0 & \overline{u}_3 & -\overline{u}_2 & u_1 & d_1 \\ & 0 & \overline{u}_1 & u_2 & d_2 \\ & & 0 & u_3 & d_3 \\ & & & 0 & e^+ \\ & & & & 0 \end{pmatrix}. \tag{2.107}$$

When we consider the asymptotic reach of the monopole, only a subset $U(1)_{\text{eff}}$ leaks out of the core. This is the effective charge group after the two spontaneous breakdowns (see Figure 3). When the group theory dust settles, and we account for the hypercharges Y and color hypercharges Y_8 of the particles, we can identify a subgroup $SU(2)_m$ as having one foot in the color $SU(3)_c$ and one foot in the ordinary $U(1)_{\text{em}}$ gauge groups as

$$SU(2)_m \hookrightarrow \begin{pmatrix} \mathbb{1}_2 & & 0 \\ & SU(2) & \\ 0 & & 1 \end{pmatrix} \tag{2.108}$$

from which we can extract the effective $U(1)_{\text{eff}}$ monopole as described above. Adding the charges gives effective charge $q_e = q_{\text{em}} + q_{colour} = (0, 0, 1, -1, 0)$ so that we assign can assign to the fermions of the model the effective charges

- $q_e = +1$ to $\overline{d}_3$, u_1, u_2, e^+ and
- $q_e = 0$ to d_1, d_2, u_3, ν_e.

And so, we only the first set of fermions *see* the $SU(5)$ monopole far away. It would seem the $SU(5)$ physics at the core is no longer relevant non-relativist fermions far away from monopole. This is wrong and we will see that the dyonic excitations left when a fermion crosses the core will be crucial in imposing the right boundary conditions.

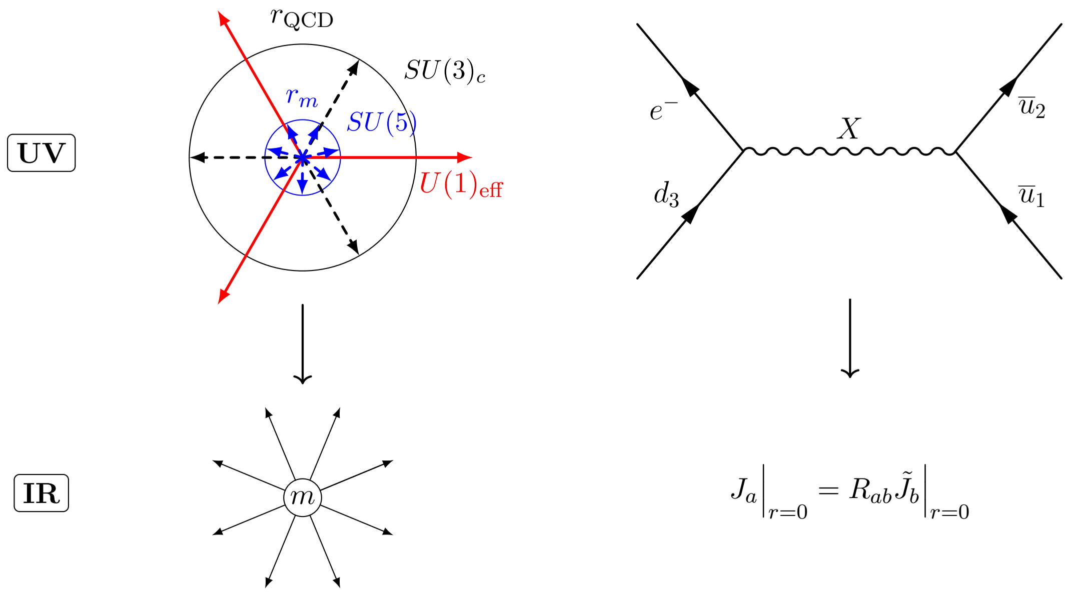

Figure 3. Far away from a non-Abelian monopole—equivalently for fermions with low enough energy—the non-Abelian gauge fields are screened enough so that it is accurate to effectively describe the core as a point-like, Dirac monopole. The structure of the core, and its baryon number violating effects, are then captured by boundary conditions in the (1+1)-dimensional effective theory.

3 Two consequences

Equipped with solutions of s-wave states in the presence of a monopole, we can ask: What are the consequences? We already saw there is a restriction on icoming/outgoing states depending on the charge and helicity of the particle. We shall see this appears to over constrain the possible outgoing states when $N \geq 4$ types of fermions are present. Another consequence will be that, because s-wave fermions don't feel the same centrifugal barrier as $j > j_0$ partial waves, they always fall on/emerge from the monopole. This guaranteed sinking/sourcing leads to unsuppressed cross sections for scattering processes which violate anomalous symmetries.

3.1 The unitarity paradox

For the sake of simplicity, let us continue assuming the fermions are massless. The case of interest is a Dirac fermion with $j = j_0$ near a point monopole, which has wave equation

$$\mathrm{i}\partial_t \Psi_{j_0,E}(\mathbf{x},t) = H\Psi_{j_0,E}(\mathbf{x},t) = \sum_{m_j} \begin{pmatrix} -(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}) & 0 \\ 0 & (\boldsymbol{\sigma}\cdot\boldsymbol{\pi}) \end{pmatrix} \frac{1}{r} \begin{pmatrix} \chi_{\text{in}}(t+r)\eta_{m_j}(\phi,\theta) \\ \chi_{\text{out}}(t-r)\eta_{m_j}(\phi,\theta) \end{pmatrix}. \qquad (3.1)$$

Using the action of helicity on s-wave spinors (2.81), this equation further reduces to

$$\mathrm{i}\partial_t \Psi_{j_0,E} = \frac{\mathrm{i}}{r}\frac{\kappa}{|\kappa|}\gamma^5 \partial_r \sum_{m_j} \begin{pmatrix} \chi_{\text{in}}(t+r)\eta_{m_j}(\phi,\theta) \\ \chi_{\text{out}}(t-r)\eta_{m_j}(\phi,\theta) \end{pmatrix}. \tag{3.2}$$

which can suggestively be written as

$$\sum_{m_j} \begin{pmatrix} \left(\mathrm{i}\partial_t - \mathrm{i}\frac{\kappa}{|\kappa|}\partial_r\right) \chi_{\text{in}}(t+r)\eta_{m_j}(\phi,\theta) \\ \left(\mathrm{i}\partial_t + \mathrm{i}\frac{\kappa}{|\kappa|}\partial_r\right) \chi_{\text{out}}(t-r)\eta_{m_j}(\phi,\theta) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.3}$$

from which we clearly glean that the angular harmonic η_{m_j} plays no role in determining the dynamics. If we group the radial, Grassmann components into a column

$$\chi(t,r) \equiv \begin{pmatrix} \chi_{\text{in}}(t+r) \\ \chi_{\text{out}}(t-r) \end{pmatrix}, \tag{3.4}$$

then we can isolate the radial equation in (3.3) as

$$\left(\partial_t - \frac{\kappa}{|\kappa|}\sigma^3\partial_r\right)\chi(r,t) = 0. \tag{3.5}$$

This looks an awful lot like the Dirac equation in two dimensions. Indeed, multiplying both sides of (3.5) by σ^1, we get $\left(\sigma^1\partial_t + \mathrm{i}\frac{\kappa}{|\kappa|}\sigma^2\partial_r\right)\chi(r,t) = 0$ such that if we make the identification

$$\tilde{\gamma}^0 = \sigma^1, \quad \tilde{\gamma}^1 = \mathrm{i}\frac{\kappa}{|\kappa|}\sigma^2, \qquad \tilde{\gamma}^5 = -\tilde{\gamma}^0\tilde{\gamma}^1 = \frac{\kappa}{|\kappa|}\sigma^3, \tag{3.6}$$

then we have found new $(1+1)$-dimensional analogues of the (chiral) gamma matrices which fulfill the 2d the Clifford algebra $\{\tilde{\gamma}^\alpha, \tilde{\gamma}^\beta\} = 2\eta^{\alpha\beta}\mathbb{1}$ for $\alpha = 0,1$, and we have recovered the Dirac equation

$$\mathrm{i}\tilde{\gamma}^\alpha\partial_\alpha\chi(t,r) = 0. \tag{3.7}$$

The problem of a massless s-wave fermion in $(3+1)$-dimension near a point monopole has thus been reduced to solving the free $(1+1)$-dimensional massless Dirac equation (3.7).

2d Weyl	4d Weyl	$U(1)$	Helicity	Direction
$\chi^+_{\text{in,a}}$	ψ^+_a	$+$	$-$	incoming
$\chi^-_{\text{in},\bar{a}}$	$\overline{\psi}^-_{\bar{a}}$	$-$	$+$	incoming
$\chi^-_{\text{out,a}}$	ψ'^-_a	$-$	$-$	outgoing
$\chi^+_{\text{out},\bar{a}}$	$\overline{\psi}'^+_{\bar{a}}$	$+$	$+$	outgoing

Table 2. The effective 2d and corresponding 4d fermions and their allowed incoming/outgoing status.

From here on out, assume $q_m > 0$ for simplicity. Consider a Dirac fermion with $q_e = +1$. Following the notation of [9], let the Dirac spinor be written in terms of two left-handed Weyl's ψ^+ and ψ'^- such that

$$\Psi_{j,E} = \begin{pmatrix} \psi^+ \\ \overline{\psi}'^+ \end{pmatrix}. \tag{3.8}$$

Then, for s-waves, we have the correspondence $\psi^+ \sim \chi^+_{\text{in}}$, $\psi'^- \sim \chi^-_{\text{out}}$ between 4d and 2d fermions. If we assume the angular momentum is conserved in a scattering, i.e. the monopole is spherically symmetric, for an incoming χ^+_{in} state there must come out a combination of $\chi^{\pm}_{\text{out}}$. Once again, we are confronted with the choice of violating helicity or $U(1)$ charge by imposing a boundary condition at the origin. Invited by the ABJ anomaly, we choose to conserve charge such that, in the effective 2d picture, the scattering of a χ^+_{in} with the monopole must be

$$\chi^+_{\text{in}} + (m) \longrightarrow \chi^+_{\text{out}} + (m). \tag{3.9}$$

In our 4d world, this corresponds to

$$\psi^+ + (m) \longrightarrow \overline{\psi}'^+ + (m). \tag{3.10}$$

This is entirely analogous to $e^+_L \to e^+_R$. Suppose now, instead of just 1 Dirac fermion, we had Ψ_a with $a = 1, ..., N$ all having $q_e = +1$. According to our discussion of $SU(2) \longrightarrow U(1)$eff, this is equivalent to N left-handed Weyl's $\psi^1_i \sim \psi^+_a$ with charges $+1$ and N $\psi^2_i \sim \psi'^-_a$ with charges -1. The 2d correspondence follows exactly as above and the situation is summarised in Table 2. In this situation, what do we get when we scatter $\chi^+_{\text{in},a}$? To understand this, we need to determine what boundary conditions must be placed at the origin.

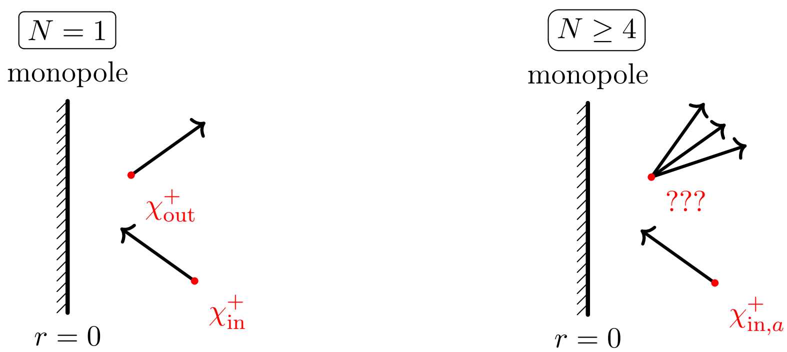

Figure 4. The dynamics of an s-wave fermion in the background of a magnetic monopole are captured by an effective $(1+1)$-dimensional free theory where the monopole leads to a boundary condition at $r = 0$. (Here the vertical dimension is time.) Unlike the $N = 1$ and $N = 2$ cases, for $N \geq 4$ fermions it seems there is no outgoing state conserving the relevant quantum numbers of the ingoing state.

Boundary conditions

In the two-dimensional picture, we can explain why the elastic scattering $e^-_R + (m) \longrightarrow e^-_R + (m)$ was not allowed. This is because the Hamiltonian, when acting on s-wave fermions $\chi(r)$, takes the form (3.2) and is not self-adjoint [14]. To cure this, recall that the conventional approach to quantum mechanics is to treat these fermionic functions $\chi(t \pm r)$ as wavefunctions in a Hilbert space which only materialise into physics via a complex inner product, i.e.

$$\chi^\dagger \psi = (\chi, \psi), \qquad |\psi|^2 = (\psi, \psi) > 0. \tag{3.11}$$

If we want H to be Hermitian *on a certain subspace* of N particles $\chi^{\pm}_i$, then we should ask

$$(\chi_{\text{in}}, H\chi_{\text{out}}) = (H\chi_{\text{in}}, \chi_{\text{out}}). \tag{3.12}$$

In particular this must hold true for a sum of such particles such that we might want to impose, noting the action of the Hamiltonian,

$$J_a\Big|_{r=0} = \tilde{J}_a\Big|_{r=0} \tag{3.13}$$

where the currents for species $\chi_{\text{in},a}$ and $\chi_{\text{out},\bar{a}}$ are given by [9, 14]

$$J_a = \sum_{\bar{a}=1}^{N} \chi^+_{\text{in},a}\chi^-_{\text{in},\bar{a}}\delta_{a,\bar{a}}, \qquad \tilde{J}_a = \sum_{\bar{a}=1}^{N} \chi^-_{\text{out},a}\chi^+_{\text{out},\bar{a}}\delta_{a,\bar{a}}. \tag{3.14}$$

This is easily solved by a $N \times N$ unitary matrix U_{ab} such that

$$\chi^+_{\text{in},a}\Big|_{r=0} = \sum_{b=1}^{N} U_{ab}\chi^-_{\text{out},b}\Big|_{r=0}. \tag{3.15}$$

It turns out these linear boundary conditions simply do not allow for an outgoing state if the ingoing is $\chi^+_{\text{in},a}$ and $N \geq 4$. We have thus come across a unitarity paradox. This is because linear (and even quadratic) relations are simply not enough to capture the monopole core effects, and more complicated must be sought. In the end, we will need to impose a dyonic boundary condition

$$J_a\Big|_{r=0} = R_{ab}\tilde{J}_b\Big|_{r=0}. \tag{3.16}$$

where the mixing is given by [11]

$$R_{ab} = \delta_{ab} - \frac{2}{N}. \tag{3.17}$$

We see why $N \geq 4$ is a particular threshold: it's the first sensible (odd N models suffer from the Witten anomaly) scenario where *fractional* fermion numbers appear in the outgoing state. To better understand why these currents make the boundary conditions simpler, let us turn to bosonisation. Whether these *semitons* [15] correspond to real particles is still disputed [16–18].

We can identify the fermions in the $N = 4$ with the $SU(5)$ effectively charged particles [7, 10], and this leads to the scattering process

$$u_1 \;+\; u_2 \;+\; m \longrightarrow \bar{d}_3 \;+\; e^+ \;+\; m. \tag{3.18}$$

Bosonisation: a new hope

It turns out [2, 7, 19, 20] that for massless chiral fermions $\psi_\pm$ in 2d, as in the discussion above, there is a correspondence with compact bosons (where $\phi \in [0, 2\pi)$ having action

$$S = \int d^2x\, \frac{1}{8\pi}\left(\partial_\alpha\phi\partial^\alpha\phi\right). \tag{3.19}$$

The correspondence such that the periodic theory allows for the identification

$$\psi_\pm(x) = \frac{1}{\sqrt{2\pi\epsilon}}e^{\mp i\phi_\pm(x)} \tag{3.20}$$

where ϵ is a UV-regulating parameter. To state this, one matches the Green's functions of the two theories. In this language, the currents J_a and $\tilde{J}_a$ correspond to simpler operators, namely the derivative of the bosons. Schematically, one has

$$J^\alpha_a \sim \overline{\chi}_a\tilde{\gamma}^\alpha\chi_a = \frac{1}{2\pi}\varepsilon^{\alpha\beta}\partial_\beta\phi_a \tag{3.21}$$

where for each species of fermion there is an associated chiral boson ϕ_a. A similar relation holds for axial currents, radial incoming currents, and fermion numbers so that these may be translated into the bosonic language for simpler interpretation [14, 21].

3.2 The Callan-Rubakov effect

Ordinarily, due to mass of X gauge bosons, processes of the type

$$u_1 \ + \ u_2 \ \longrightarrow \ X \ \longrightarrow \ \overline{d}_3 \ + \ e^+ \tag{3.22}$$

are an s-channel process with scattering cross section scaling as $\sigma \sim 1/(s - m_X^2) \sim 1/m_X^2$, which is tiny for a GUT such as $SU(5)$. However, in the presence of a monopole the process

$$u_1 \ + \ \textcircled{m} \ \longrightarrow \ \frac{1}{2}\left(\overline{u}_2 + \overline{d}_3 + e^-\right) \ + \ \textcircled{m} \tag{3.23}$$

has cross section scaling like $\sigma \sim 1/s$ instead [10]. We see that, by involving itself in the scattering process, the monopole has removed the grand suppression by a superheavy mass. As a result the probability of this version of the process is much greater, so we say the monopole has *catalysed* the process.

We saw that monopoles induce chirality-violating processes in Section 2. We now also see that it catalyses baryon-violating processes. The first one is not a surprise because of the chiral anomaly (in this case the suppression in the absence of monopoles is by the instanton tunnelling factor $e^{-8\pi^2 n/g^2}$ [7, 19]). The second is not a surprise either because the Dirac monopole is supposedly the limit of a GUT monopole in which heavy boson exchange can carry away baryon number. In both cases, there is a symmetry which we know to be broken in the quantum theory and the monopole catalyses this violation: this is the Callan-Rubakov effect. (One could also view the semitonic decay as the Callan-Rubakov effect for species number, which is not a good quantum symmetry [11].)

4 Conclusion

$$\begin{tikzcd}
\text{ABJ anomaly} \arrow[r, Rightarrow] & \text{Helicity violated} \arrow[dr, Rightarrow, bend left] & & \text{Unitarity paradox} \\
\text{Monopole} \arrow[u, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Boundary conditions} \arrow[r, Rightarrow] & s\text{-wave is special} \arrow[ur, Rightarrow, bend right] \arrow[dr, Rightarrow, bend left] & \\
\mathbf{J} \sim -\kappa\hat{\mathbf{r}} \arrow[r, Rightarrow] & (\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\eta_{m_j} = \mathrm{sgn}(\kappa)\eta_{m_j} \arrow[ur, Rightarrow, bend right] & & \text{Callan-Rubakov effect}
\end{tikzcd}$$

Figure 5. When a massless s-wave fermion is in the background of a magnetic monopole, the monopole field leads to a chiral anomaly and an additional term $-\kappa\hat{\mathbf{r}}$. These cause a butterfly effect which climaxes in the unitarity paradox and the Callan-Rubakov effect.

We saw that, in the background of an Abelian monopole field, there is no outgoing state corresponding to, say, an e_R^- in (toy model) *unless* one allows for semiton states made of particles with fractional quantum numbers. The interpretation of this bizarre final state in 4d is an open problem [9, 16–18, 22].

In light of the difficulty in interpreting the semiton, alternative resolutions have been proposed for the unitarity paradox. For example, the authors of [16] claim that the jump we made from $(3+1)$-dimensions to the effective $(1+1)$-dimensional picture was not comprehensive in that important features are truncated. These features produce the 4d scattering process $u + \textcircled{m} \longrightarrow \overline{u} + \overline{d} + e^+ + \textcircled{m}$ in which—despite the outgoing $\overline{u}$ not being possible as an s-wave state by virtue of (2.82), and thus being a higher angular momentum mode—the final *entangled* product has $j = j_0$. Other alternatives include [23] for which alternative operators are used to interpolate between incoming/outgoing modes.

The other surprising feature we saw when s-wave fermions scatter off monopoles is the Callan-Rubakov effect, which promises that certain decays are accelerated in comparison to their counterparts *without* a monopole involved. On one hand, this effect provides concrete examples of decays which violate anomalous symmetries in the quantum theory and, on the other hand, implies pretty dramatic phenomenological consequence[8]. Resolving this issue is intricately related with cosmic inflation.

We have learned two important things here. First, the s-wave dynamics of fermion-monopole scattering clearly showcase some fascinating features of monopoles and gauge theory. The second is to always check your dart before picking it off the board. You never know what prickly subtleties await.

A Monopole harmonics

For the generalised angular momentum (2.19), we seek eigenfunctions $Y_{\ell m\kappa}(\phi,\theta)$ obeying the usual

$$\mathbf{L}^2 Y_{\ell m\kappa} = \ell(\ell+1)Y_{\ell m\kappa}, \qquad L_z Y_{\ell m\kappa} = mY_{\ell m\kappa}, \tag{A.1}$$

$$L_\pm Y_{\ell m\kappa} = \sqrt{j(j+1)-m(m\pm1)}Y_{\ell(m\pm1)\kappa}. \tag{A.2}$$

If $\kappa = 0$, the above just defines the usual spherical harmonics one comes across in classical field theory and in solving the hydrogen atom. Because $\mathbf{L}$ depends on the region in which we define the potential $\mathbf{A}$, so will the monopole harmonics. Let us restrict our calculation to region R_a such that $A = ...$, noting that we can retrieve those in R_b via (2.13) [24]. A simple, if tedious, calculation yields

$$L_x = \left(\mathrm{i}\cos\phi\cot\theta\partial_\phi + \mathrm{i}\sin\phi\partial_\theta - \kappa\cos\phi\frac{1-\cos\theta}{\sin\theta}\right), \tag{A.3}$$

$$L_y = \left(\mathrm{i}\sin\phi\cot\theta\partial_\phi - \mathrm{i}\cos\phi\partial_\theta - \kappa\sin\phi\frac{1-\cos\theta}{\sin\theta}\right), \tag{A.4}$$

$$L_z = (-\mathrm{i}\partial_\phi - \kappa) \tag{A.5}$$

so that the total orbital angular momentum $\mathbf{L}^2$ and ladder operators $L_\pm = L_x \pm \mathrm{i}L_y$ are given by

$$\mathbf{L}^2 = -\frac{1}{\sin^2\theta}\left[\sin\theta\partial_\theta\left(\sin\theta\partial_\theta\right) + \left(\partial_\phi - \mathrm{i}\kappa(1-\cos\theta)\right)^2\right] + \kappa^2 \tag{A.6}$$

$$L_\pm = e^{\pm\mathrm{i}\phi}\left[\pm\partial_\theta + \mathrm{i}\cot\theta\partial\phi - \kappa\frac{1-\cos\theta}{\sin\theta}\right]. \tag{A.7}$$

Looking at (A.1), we can separate variables as $Y_{\ell m\kappa}(\phi,\theta) = e^{\mathrm{i}(m+\kappa)\phi}\Theta_{\ell m\kappa}(\cos\theta)$. A useful strategy will be to start by finding the lowest weight eigenfunction $Y_{\ell(-\ell)\kappa}$ and then repeatedly apply the raising operator L_+. Equation (A.1) implies

$$\left[\ell(\ell+1)-\kappa^2\right]\Theta_{\ell m\kappa}(\cos\theta) = \left[-\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right) + \frac{1}{\sin^2\theta}\left(m+\kappa\cos\theta\right)^2\right]\Theta_{\ell m\kappa}(\cos\theta). \tag{A.8}$$

If we make the typical substitution $x = \cos\theta$, this differential equation becomes

$$0 = \Theta''_{\ell m\kappa}(x) - \frac{2x}{(1-x^2)}\Theta'_{\ell m\kappa}(x) - \frac{1}{(1-x^2)^2}\left[(m+x\kappa)^2 - \left(1-x^2\right)\left(\ell(\ell+1)-\kappa^2\right)\right]\Theta_{\ell m\kappa}(x). \tag{A.9}$$

The differential equation relevant to solving for $Y_{\ell(-\ell)\kappa}$ takes the form

$$0 = \Theta''_{\ell(-\ell)\kappa}(x) - \frac{2x}{(1-x^2)}\Theta'_{\ell(-\ell)\kappa}(x) - \frac{1}{(1-x^2)^2}\left[(x\ell-\kappa)^2 - (1-x^2)\ell\right]\Theta_{\ell(-\ell)\kappa}(x). \tag{A.10}$$

The natural ansatz $\Theta_{\ell(-\ell)\kappa}(x) = A_{\ell\kappa}(1+x)^\alpha(1-x)^\beta$ then translates the above to

$$\begin{aligned} 0 = {}& \alpha(\alpha-1)(1-x)^2 + \beta(\beta-1)(1+x)^2 - 2\alpha\beta(1+x)(1-x) - 2x\alpha(1-x) - 2x\beta(1+x) \\ & - (x\ell-\kappa)^2 + (1-x)(1+x)\ell \end{aligned} \tag{A.11}$$

and evaluating at $x = \pm1$ gives two solutions for each exponent;

$$\alpha = \pm\frac{\ell+\kappa}{2}, \qquad \beta = \pm\frac{\ell-\kappa}{2}. \tag{A.12}$$

To be able to normalise the monopole harmonics as

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \ \sin\theta \, |Y_{\ell m\kappa}|^2 = 1, \tag{A.13}$$

which becomes

$$2\pi \, |A_{\ell\kappa}|^2 \int_{-1}^{+1} dx \, (1+x)^{2\alpha} \, (1-x)^{2\beta} = 1, \tag{A.14}$$

both exponents α and β must be positive for convergence. Since we need the monopole harmonics to be single-valued on each region, such that $Y_{\ell m\kappa}(\phi + 2\pi, \theta) = Y_{\ell m\kappa}(\phi, \theta)$, the angular momentum projection satisfies $m + \kappa \in \mathbb{Z}$. But $2\kappa \in \mathbb{Z}$ by the Dirac quantisation condition, which immediately implies $\ell \pm \kappa \in \mathbb{Z}$. Noting the ranges (2.34) & (2.35), this means $\ell \pm \kappa \in \mathbb{Z}_{\geq 0}$ such for $\alpha, \beta > 0$, we need $\ell \geq |\kappa|$. The normalisation (A.14) can easily be written as a beta function and results in the overall normalisation

$$A_{\ell\kappa} = 2^{-\ell} \sqrt{\frac{(2\ell+1)!}{4\pi(\ell+\kappa)!(\ell-\kappa)!}} \tag{A.15}$$

such that the lowest weight eigenfunction is

$$\Theta_{\ell(-\ell)\kappa}(x) = A_{\ell\kappa} \sqrt{1+x}^{\ell+\kappa} \sqrt{1-x}^{\ell-\kappa}. \tag{A.16}$$

One can then prove (2.33) by induction, applying L_+. Note

$$x = \cos\theta, \qquad N_{\ell m\kappa} = (-1)^{\ell+m} 2^{-\ell} \sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell-\kappa)!(\ell+\kappa)!(\ell+m)!}}. \tag{A.17}$$

Next, we solve for the coefficients B_{ij} in (2.53) by studying the $m_j = -j$ eigenfunctions. These are

$$\mathcal{Y}^{(1)}_{j(-j)\kappa} = \begin{pmatrix} 0 \\ Y_{(j-\frac{1}{2})(-j+\frac{1}{2})\kappa} \end{pmatrix} \equiv \begin{pmatrix} 0 \\ Y_{①} \end{pmatrix}, \quad \mathcal{Y}^{(2)}_{j(-j)\kappa} = \begin{pmatrix} -\sqrt{\frac{2j+1}{2j+2}} Y_{(j+\frac{1}{2})(-j-\frac{1}{2})\kappa} \\ \frac{1}{\sqrt{2j+2}} Y_{(j+\frac{1}{2})(-j+\frac{1}{2})\kappa} \end{pmatrix} \equiv \begin{pmatrix} -\sqrt{\frac{2j+1}{2j+2}} Y_{②} \\ \frac{1}{\sqrt{2j+2}} Y_{③} \end{pmatrix} \tag{A.18}$$

where, assuming $j > j_0$ so that $\mu > 0$,

$$Y_{①} = 2^{-j+\frac{1}{2}} \sqrt{\frac{(2j)!}{4\pi(j+\frac{1}{2}-\kappa)!(j+\frac{1}{2}+\kappa)!}} e^{\mathrm{i}(\kappa-j+\frac{1}{2})\phi} \sqrt{1+x}^{j-\frac{1}{2}+\kappa} \sqrt{1-x}^{j-\frac{1}{2}-\kappa} \Big|_{x=\cos\theta}, \tag{A.19}$$

$$Y_{②} = \frac{\sqrt{(2j+2)(2j+1)}}{2\mu} e^{-\mathrm{i}\phi} \sin\theta Y_{①}, \qquad Y_{③} = (-1) \frac{\sqrt{2j+2}}{2\mu} ((2j+1)\cos\theta - 2\kappa) Y_{①}. \tag{A.20}$$

Solving the equations

$$(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}) \mathcal{Y}^{(1)}_{j(-j)\kappa} = \begin{pmatrix} -B_{12} \sqrt{\frac{2j+1}{2j+2}} Y_{②} \\ B_{11} Y_{①} + B_{12} \frac{1}{\sqrt{2j+2}} Y_{③} \end{pmatrix}, \qquad (\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}) \mathcal{Y}^{(2)}_{j(-j)\kappa} = \begin{pmatrix} -B_{22} \sqrt{\frac{2j+1}{2j+2}} Y_{②} \\ B_{21} Y_{①} + B_{22} \frac{1}{\sqrt{2j+2}} Y_{③} \end{pmatrix} \tag{A.21}$$

using the explicit form (2.62) then gives (2.54), as in [24]. Turning our attention to $j = j_0$, we calculate the explicit form

$$\eta_{-j_0} = -\frac{2^{-|\kappa|}}{\sqrt{8\pi|\kappa|}} e^{\mathrm{i}(\kappa-|\kappa|)} (\sin\theta)^{|\kappa|} \left(\frac{1+\cos\theta}{1-\cos\theta} \right)^{\frac{\kappa}{2}} \begin{pmatrix} 1 \\ e^{\mathrm{i}\phi} \frac{\kappa/|\kappa| - \cos\theta}{\sin\theta} \end{pmatrix} = \text{factor} \times \begin{pmatrix} 1 \\ e^{\mathrm{i}\phi} \frac{\kappa/|\kappa| - \cos\theta}{\sin\theta} \end{pmatrix} \tag{A.22}$$

which clearly becomes (2.82) when multiplied by $(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}})$.

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